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R. C. J. Howland and R. C. Knight

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# STRESS FUNCTIONS FOR A PLATE CONTAINING GROUPS OF CIRCULAR HOLES

*By* R. C. J. HOWLAND, M.A., D.Sc.
*Late Professor of Mathematics, University College, Southampton*

AND R. C. KNIGHT, M.Sc., Ph.D.
*King's College, London*

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## Introduction

A number of investigations, both experimental and theoretical, have been made to determine how the presence of holes in a uniform plate under given applied forces effects the distribution of the stresses in the plate (see Coker and Filon 1931, chap. IV). When there is a single hole in a plate which may be considered infinite, the problem is elementary; but a hole near to a straight boundary or to a similar hole greatly influences the maximum stress and complicates the mathematical solution. No general method of solution has been given and we now extend methods, previously used by the present writers in particular cases, to a group of problems in which the boundaries possess a certain invariance. The boundaries we shall consider are a set of equal circles together with in some cases a pair of parallel straight lines. With each of the circles is associated a rectangular co-ordinate system, and it is essential to the method that the boundaries, boundary conditions and infinity conditions should remain invariant under a transformation in which each co-ordinate system and corresponding circle transforms into another system and circle of the set.

The following configurations have boundaries with which we can deal:

(1) One pair of circles.

(2) Two pairs of circles.

(3) A single infinite row of circles.

(4) A double infinite row of circles with arbitrary stagger.

(5) A doubly periodic distribution of circles.

(6) One circle, a pair of circles, or an infinite row of circles with one straight boundary similarly related to all the circles.

(7) One circle between parallel lines.

(8) One pair or two pairs of circles symmetrically placed between parallel lines.

(9) A single row of circles symmetrically placed between parallel lines.

Of these (1) may be treated using bipolar co-ordinates, which method has been used by Jeffery (1920) to obtain a solution for the first of (6). A solution for (3) has recently been published by one of the authors (Howland 1930). (7) has been discussed by us in previous papers (e.g. Howland and Stevenson 1933; Knight 1934). Doubly periodic functions are required for cases (5) and (9), and, as they introduce considerations of rather a different character, will not be dealt with here. The second and third of (6) reduce in special cases to (2) or (4) and, generally, are better treated in a different manner. This leaves (2), (4), (8) to be considered and (1) will be added to these for the sake of comparison.

Our method consists in constructing biharmonic stress functions which are invariant under the same transformation as that which transforms one circle into another of the set. Then the functions are expanded about the centre of one of the circles. If the boundary conditions, which also must remain invariant, are now satisfied on this circle using these functions, the conditions on the other circles of the set will be automatically satisfied. When there are additional straight line boundaries the functions must be constructed so that the conditions on these are satisfied in advance.

It will be noticed that the transformations which leave the boundaries unaltered are of the type

$$\mathrm{T}_1, \quad z' = z+a, \qquad \mathrm{T}_3, \quad z' = -z+c,$$

$$\mathrm{T}_2, \quad z' = \bar{z}+b, \qquad \mathrm{T}_4, \quad z' = -\bar{z}+d,$$

where $z$ and $z'$ are complex variables associated with the centre of two of the circles and $a$, $b$, $c$, $d$ are, in general, complex quantities depending on the particular configuration considered. The stress function must then be of the form

$$\chi(z) = \Sigma f(\mathrm{T}z),$$

where T is one of the transformations of the group.

## 1. The general method of solution

Consider a uniform plate of infinite extent or bounded by parallel straight lines and containing a number of equal circular holes having an invariancy of the type mentioned. It is supposed to be in a state of generalized plane stress defined by a stress function $\chi$. This function must be biharmonic, i.e. it must be a solution of the equation

$$\nabla^4\chi = 0. \tag{1·1}$$

The boundary conditions to be satisfied are:

(*a*) the stresses $\widehat{xx} \equiv \frac{\partial^2 \chi}{\partial y^2}, \quad \widehat{yy} \equiv \frac{\partial^2 \chi}{\partial x^2}, \quad \widehat{xy} \equiv -\frac{\partial^2 \chi}{\partial x \partial y}$

must tend to constant values at infinity;

(*b*) on the circumference of each circular hole, the stresses

$$\widehat{rr} \equiv \frac{1}{r^2}\frac{\partial^2 \chi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \chi}{\partial r} = 0, \quad \widehat{r\theta} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \chi}{\partial \theta}\right) = 0;$$

(*c*) if there are straight boundaries parallel to the *x*-axis, the stresses $\widehat{yy}$ and $\widehat{xy}$ must vanish on them.

We first construct a set of biharmonic functions having singularities at the centres of the circles and which are invariant under the transformation which leaves the circles unchanged. If there are straight boundaries the condition (*c*) must be satisfied as well. Two methods may be conveniently employed to obtain these functions.

When the plate is infinite we start with a complex harmonic function

$$w_0(z) = u_0 - iv_0, \tag{1·2}$$

having the requisite singularities and invariance. Differentiation of $w_0(z)$ $s$ times leads to the function

$$w_s(z) = u_s - iv_s. \tag{1·3}$$

The real and imaginary parts of the complex functions defined by equations (1·2) and (1·3) will be functions satisfying the conditions, and have the form

$$\left.\begin{aligned} u_0 &= -\log r + \sum_{n=1}^{\infty} r^n ({}^0\alpha_n \cos n\theta + {}^0\beta_n \sin n\theta), \\ v_0 &= \theta + \sum_{n=1}^{\infty} r^n ({}^0\gamma_n \cos n\theta + {}^0\delta_n \sin n\theta), \end{aligned}\right\} \tag{1·4}$$

$$\left.\begin{aligned} u_s &= r^{-s} \cos s\theta + \sum_{n=1}^{\infty} r^n ({}^s\alpha_n \cos n\theta + {}^s\beta_n \sin n\theta), \\ v_s &= r^{-s} \sin s\theta + \sum_{n=1}^{\infty} r^n ({}^s\gamma_n \cos n\theta + {}^s\delta_n \sin n\theta). \end{aligned}\right\} \tag{1·5}$$

In equations (1·5) $s$ can take all integral values $>0$, and we therefore have a double infinity of suitable harmonic functions. The coefficients ${}^0\alpha_n$, etc. depend upon the particular problem considered in which the polar co-ordinates used have their pole at the centre of one of the circles of the system. From these functions we next construct biharmonic functions satisfying the same conditions. The method of obtaining them varies according to the problem to be solved, but they will be of the type

$$\left.\begin{aligned} \mathrm{U}_s &= r^{-s+2} \cos s\theta + \sum_{n=0}^{\infty} r^n \{({}^s\mathrm{A}_n + {}^s\mathrm{A}'_n r^2) \cos n\theta + ({}^s\mathrm{B}_n + {}^s\mathrm{B}'_n r^2) \sin n\theta\}, \\ \mathrm{V} &= r^{-s+2} \sin s\theta + \sum_{n=0}^{\infty} r^n \{({}^s\mathrm{C}_n + {}^s\mathrm{C}'_n r^2) \cos n\theta + ({}^s\mathrm{D}_n + {}^s\mathrm{D}'_n r^2) \sin n\theta\}. \end{aligned}\right\} \tag{1·6}$$

These functions will usually be sufficient for our purpose (see Howland 1935) of finding the stress function for the problem. In certain cases further functions may be required. They will be given later when the various configurations of circles are considered. They are found by considering the stress functions for forces acting, in suitable directions, at the centres of the circles. Leaving, for the moment, the method of constructing the functions when the plate is of finite breadth, we next have to combine the stress functions given above in such a way that the boundary conditions (*b*) on the circles are satisfied. The functions are such that if the conditions are satisfied on one circle they will be automatically satisfied on each of the others. In each case considered below we shall construct all the functions sufficient to solve the most general problem, but to indicate the method of solution we assume here some simplification. Suppose symmetry conditions are such that our functions are even in both $x$ and $y$. Our coefficients are then all zero except those of the even cosines and terms independent of $\theta$. Suppose further that $\chi_0$ is a stress function which would give the stresses in the plate if the holes were not present, i.e. the stress function giving the infinity conditions. We then assume as our final stress function

$$\chi = \chi_0 + L_0 u_0 + \sum_{s=1}^{\infty} (L_{2s} u_{2s} + M_{2s} U_{2s}). \tag{1·7}$$

The constants L, M have to be chosen so that the conditions (*b*) are satisfied. Hence we substitute this $\chi$ into the equations for $\widehat{rr}$ and $\widehat{r\theta}$ and equate to zero after putting $r = a$ (the radius of the circle). This leads to a double infinity of linear equations between the constants. They are of the form

$$\left.\begin{aligned} L_{2n} &= P_{2n} + {}^{2n}h_0 L_0 + \sum_{s=1}^{\infty} ({}^{2n}h_{2s} L_{2s} + {}^{2n}i_{2s} M_{2s}), \\ M_{2n} &= Q_{2n} + {}^{2n}j_0 L_0 + \sum_{s=1}^{\infty} ({}^{2n}j_{2s} L_{2s} + {}^{2n}k_{2s} M_{2s}), \end{aligned}\right\} \tag{1·8}$$

where $P_{2n}$, $Q_{2n}$ are known constants depending on $\chi_0$, and the new coefficients ${}^{2n}h_{2s}$, etc. depend upon the coefficients ${}^{2s}A_{2n}$, etc., and upon $a$. A formal solution of the equations is given by

$$L_{2n} = \sum_{r=0}^{\infty} L_{2n}^{(r)}, \quad M_{2n} = \sum_{r=0}^{\infty} M_{2n}^{(r)}, \tag{1·9}$$

where

$$\left.\begin{aligned} L_{2n}^{(0)} &= P_{2n}, \quad M_{2n}^{(0)} = Q_{2n}, \\ L_{2n}^{(r)} &= {}^{2n}h_0 L_0^{(r-1)} + \sum_{s=1}^{\infty} ({}^{2n}h_{2s} L_{2s}^{(r-1)} + {}^{2n}i_{2s} M_{2s}^{(r-1)}), \\ M_{2n}^{(r)} &= {}^{2n}j_0 L_0^{(r-1)} + \sum_{n=1}^{\infty} ({}^{2n}j_{2s} L_{2s}^{(r-1)} + {}^{2n}k_{2s} M_{2s}^{(r-1)}). \end{aligned}\right\} \tag{1·10}$$

The validity of this solution is established if we prove that the series in (1·9) and (1·10) are convergent. This can be done in particular cases if suitable inequalities are found for the coefficients ${}^{2s}A_{2n}$, etc. In every case so far investigated the series have been proved convergent provided the ratio of the radius of a circle to some other typical length, such as the distance between two centres, is not too large (see e.g. Knight 1934).

For the problems in which the plate is of finite breadth the sets of stress functions are most conveniently found by a different method. We start with a fundamental stress function representing forces at the centres of the circles which leave the straight edges free from stress. This is expanded in a series about one centre. The coefficients in these series contain $\sigma$, the modified value of Poisson's ratio used in generalized plane stress (Filon 1903, p. 67; Love 1927, p. 138). These functions are biharmonic functions for all values of $\sigma$, and this may be treated as an arbitrary constant provided the functions obtained from the fundamental one are of the type giving single-valued displacements (Knight 1934, p. 256). Having given $\sigma$ a particular value we obtain by successive differentiation functions of type (1·6).

In the most general problem four sets of functions will be required, two sets $u_s$ and two $v_s$. These may be found from two fundamental functions by giving $\sigma$ two special, suitable values. The method of solution is the same as before, for the infinite plate; but it is, in general, complicated by the fact that four sets of unknown coefficients will have to be evaluated by the successive approximations.

## 2. Two equal circles in an infinite plane

Let the co-ordinate systems associated with the circles as shown in fig. 1 be defined by

$$z = x + iy, \quad z' = x' + iy'. \tag{2·1}$$

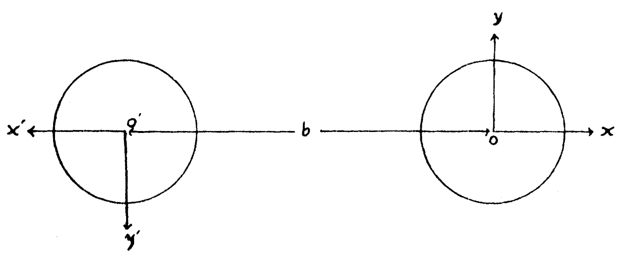


Fig. 1

We then have invariance for reflexion in a line parallel to the $y$ axis midway between the centres. If the centres are at a distance $b$ apart, the relation between $z$ and $z'$ is

$$z' = -(\bar{z} + b).$$

Define subsidiary variables by

$$\left.\begin{aligned} \zeta = z/b, \quad \zeta' = z'/b = -(\bar{\zeta}+1), \\ \xi = x/b, \quad \eta = y/b, \quad \rho e^{i\theta} = \zeta. \end{aligned}\right\} \tag{2·2}$$

Harmonic functions of the required type are

$$\begin{aligned} -w_0 &= \log \zeta + \log \zeta', \\ w_s &= \frac{(-)^{s-1}}{(s-1)!}\left\{\frac{d^s}{d\zeta^s}(\log \zeta) + \frac{d^s}{d\zeta'^s}(\log \zeta')\right\} \\ &= \zeta^{-s} + (-)^s (1+\bar{\zeta})^{-s}, \quad (s \geqslant 1). \end{aligned}$$

Omitting a constant from $w_0$ which is trivial, we have the following expansions:

$$w_0 = -\log \zeta + \sum_{n=1}^{\infty} \frac{(-)^n}{n} \zeta^n, \tag{2·3}$$

$$w_s = \zeta^{-s} + \sum_{n=0}^{\infty} (-)^{n+s} \binom{n+s-1}{n} \zeta^n. \tag{2·4}$$

If we write $w_s = u_s - iv_s$, we obtain, by equating real and imaginary parts, the functions

$$\left.\begin{aligned} u_0 &= -\log \rho + \sum_{n=1}^{\infty} \frac{(-)^n}{n} \rho^n \cos n\theta, \\ v_0 &= \theta + \sum_{n=1}^{\infty} \frac{(-)^n}{n} \rho^n \sin n\theta, \end{aligned}\right\} \tag{2·5}$$

$$\left.\begin{aligned} u_s &= \rho^{-s} \cos s\theta + \sum_{n=0}^{\infty} (-)^{n+s} \binom{n+s-1}{n} \rho^n \cos n\theta, \\ v_s &= \rho^{-s} \sin s\theta + \sum_{n=0}^{\infty} (-)^{n+s} \binom{n+s-1}{n} \rho^n \sin n\theta. \end{aligned}\right\} \tag{2·6}$$

Suitable biharmonic functions are obtained from these by writing, when $s > 2$,

$$U_s = u_{s-2} - 2\eta v_{s-1}, \quad V_s = v_{s-2} + 2\eta u_{s-1}.$$

Their expansions are

$$\left.\begin{aligned} U_s &= \rho^{-s+2} \cos s\theta + \sum_{n=0}^{\infty} ({}^sA_n + {}^sA'_n \rho^2) \rho^n \cos n\theta, \\ V_s &= \rho^{-s+2} \sin s\theta + \sum_{n=0}^{\infty} ({}^sD_n + {}^sD'_n \rho^2) \rho^n \sin n\theta, \end{aligned}\right\} \tag{2·7}$$

where

$$\left.\begin{aligned} {}^sA_n &= {}^sD_n = (-)^{n+s} \binom{n+s-2}{n}, \\ {}^sA'_n &= {}^sD'_n = (-)^{n+s-1} \binom{n+s-1}{n+1}. \end{aligned}\right\} \tag{2·8}$$

When $s = 2$ we may take

$$\begin{aligned} U_2 &= 1 - 2\eta v_1 \\ &= \cos 2\theta + \sum_{n=0}^{\infty} (-)^n (1-\rho^2) \rho^n \cos n\theta, \\ V_2 &= 2\eta u_1 \\ &= \sin 2\theta + \sum_{n=0}^{\infty} (-)^n (1-\rho^2) \rho^n \sin n\theta. \end{aligned}$$

Thus the case $s = 2$ is not special and the coefficients in the series may be obtained by putting $s = 2$ in the equations (2·8).

The biharmonic functions $U_1$ and $V_1$ will not be required since they correspond to zero stresses, but functions $U_0$ and $V_0$ may be required in any particular problem. Some care is required in these cases in order that the displacements obtained from them may be single-valued.

Such a stress function is that for an isolated force acting at the origin. If the force acts in direction $\overrightarrow{OX}$, we have, apart from a multiplying factor,

$$\chi_1(\zeta) = \frac{1-2\sigma}{1-\sigma}\xi\log\rho - 2\eta\theta, \tag{2·9}$$

where $\sigma$ is the modified Poisson's ratio.

Hence we define

$$\begin{aligned} U_0 &= \chi_1(\zeta)+\chi_1(\zeta') \\ &= \frac{1-2\sigma}{1-\sigma}\rho\cos\theta\log\rho - 2\theta\rho\sin\theta + \sum_{n=0}^{\infty}({}^0A_n + {}^0A'_n\rho^2)\,\rho^n\cos n\theta. \end{aligned} \tag{2·10}$$

Similarly, if we take an isolated force in direction $\overrightarrow{OY}$ which is

$$\chi_2(\zeta) = \frac{1-2\sigma}{1-\sigma}\eta\log\rho + 2\xi\theta, \tag{2·11}$$

we have $\quad V_0 = \chi_2(\zeta)+\chi_2(\zeta')$

$$= \frac{1-2\sigma}{1-\sigma}\rho\sin\theta\log\rho + 2\theta\rho\cos\theta + \sum_{n=0}^{\infty}({}^0D_n + {}^0D'_n\rho^2)\,\rho^n\sin n\theta. \tag{2·12}$$

In the above series the coefficients have the values

$$\left.\begin{aligned} {}^0A_n &= \frac{(-)^n}{1-\sigma}\left\{\frac{1-2\sigma}{n} - \frac{3-4\sigma}{2(n-1)}\right\}, \quad (n>1), \\ {}^0A'_n &= {}^0D'_n = \frac{(-)^n}{1-\sigma}\frac{1}{2(n+1)}, \\ {}^0D_n &= \frac{(-)^n}{1-\sigma}\left\{\frac{3-4\sigma}{2(n-1)} - \frac{2(1-\sigma)}{n}\right\}, \quad (n>1). \end{aligned}\right\} \tag{2·13}$$

The undefined coefficients may be taken to be zero.

## 3. Two pairs of circles in an infinite plane

Let the centres of the four circles be at the points given by $z = \pm p \pm iq$, where $z = x+iy$ defines a rectangular co-ordinate system. At the four centres we take four subsidiary co-ordinate systems as shown in fig. 2. We then have invariance for reflexions in the axis of $x$ and $y$. The four systems associated with the circles are

$$z_1 = x_1+iy_1 = z-p-iq, \qquad z_3 = x_3+iy_3 = \bar{z}-p-iq,$$
$$z_2 = x_2+iy_2 = -\bar{z}-p-iq, \qquad z_4 = x_4+iy_4 = -z-p-iq.$$

The complex harmonic functions given previously (Howland and McMullen 1936) are defined by

$$-w_0 = \log z_1 z_2 z_3 z_4, \tag{3·1}$$

$$w_s = \frac{(-)^s}{(s-1)!} \sum_{p=1}^{4} \frac{d^s}{dz_p^s} (\log z_p), \quad (s > 0). \tag{3·2}$$

When expanded these are

$$w_0 = -\log z_1 + \sum_{n=1}^{\infty} \frac{(-)^n}{n} \frac{\bar{z}_1^n}{(2p)^n} + \sum_{n=1}^{\infty} \frac{1}{n} \frac{\bar{z}_1^n}{(2iq)^n} + \sum_{n=1}^{\infty} \frac{(-)^n}{n} \frac{z_1^n e^{ni\phi}}{(2l)^n},$$

$$w_s = \frac{1}{z_1^s} + \sum_{n=1}^{\infty} \binom{n+s-1}{s-1} \left\{ \frac{(-)^{n+s} \bar{z}_1^n}{(2p)^{n+s}} + \frac{(-)^s \bar{z}_1^n}{(2iq)^{n+s}} + \frac{(-)^{n+s} e^{-(n+s)i\phi}}{(2l)^{n+s}} \right\},$$

in which

$$p + iq = le^{i\phi}. \tag{3·3}$$

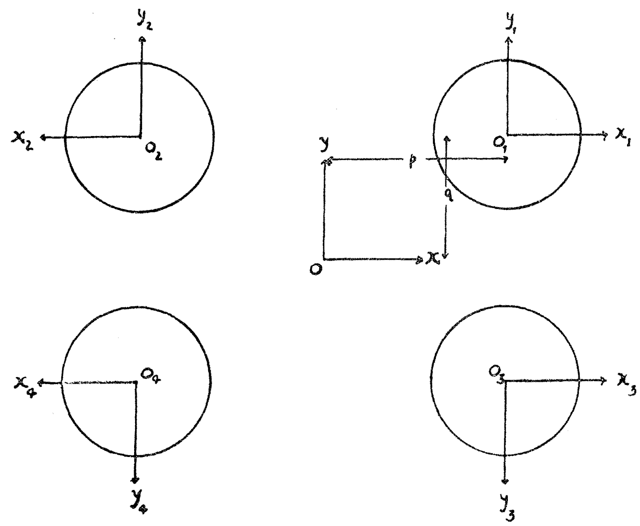


FIG. 2

If we write as before $w_s = u_s - iv_s$ and $z_1 = re^{i\theta}$,

then

$$\left.\begin{aligned} u_0 &= -\log r + \sum_{n=1}^{\infty} r^n ({}^0\alpha_n \cos n\theta + {}^0\beta_n \sin n\theta), \\ v_0 &= \theta + \sum_{n=1}^{\infty} r^n ({}^0\gamma_n \cos n\theta + {}^0\delta_n \sin n\theta), \end{aligned}\right\} \tag{3·4}$$

$$\left.\begin{aligned} u_s &= r^{-s} \cos s\theta + \sum_{n=1}^{\infty} r^n ({}^s\alpha_n \cos n\theta + {}^s\beta_n \sin n\theta), \\ v_s &= r^{-s} \sin s\theta + \sum_{n=1}^{\infty} r^n ({}^s\gamma_n \cos n\theta + {}^s\delta_n \sin n\theta); \end{aligned}\right\} \tag{3·5}$$

where the coefficients have the following alternative forms according as $n+s$ is even or odd:

$n+s$ even:

$$\left.\begin{aligned} {}^s\alpha_n &= \binom{n+s-1}{n}\left\{\frac{1}{(2p)^{n+s}}+\frac{(-)^{\frac{1}{2}(s-n)}}{(2q)^{n+s}}+\frac{\cos(n+s)\,\phi}{(2l)^{n+s}}\right\},\\ {}^s\beta_n &= {}^s\gamma_n = \binom{n+s-1}{n}\frac{\sin(n+s)\,\phi}{(2l)^{n+s}},\\ {}^s\delta_n &= \binom{n+s-1}{n}\left\{\frac{1}{(2p)^{n+s}}+\frac{(-)^{\frac{1}{2}(s-n)}}{(2q)^{n+s}}-\frac{\cos(n+s)\,\phi}{(2l)^{n+s}}\right\}; \end{aligned}\right\} \quad (3\cdot6)$$

$n+s$ odd:

$$\left.\begin{aligned} {}^s\alpha_n &= -\binom{n+s-1}{n}\left\{\frac{1}{(2p)^{n+s}}+\frac{\cos(n+s)\,\phi}{(2l)^{n+s}}\right\},\\ {}^s\beta_n &= \binom{n+s-1}{n}\left\{\frac{(-)^{\frac{1}{2}(s-n-1)}}{(2q)^{n+s}}-\frac{\sin(n+s)\,\phi}{(2l)^{n+s}}\right\},\\ {}^s\gamma_n &= -\binom{n+s-1}{n}\left\{\frac{(-)^{\frac{1}{2}(s-n-1)}}{(2q)^{n+s}}+\frac{\sin(n+s)\,\phi}{(2l)^{n+s}}\right\},\\ {}^s\delta_n &= \binom{n+s-1}{n}\left\{-\frac{1}{(2p)^{n+s}}+\frac{\cos(n+s)\,\phi}{(2l)^{n+s}}\right\}. \end{aligned}\right\} \quad (3\cdot7)$$

The coefficients in (3·4), i.e. when $s=0$, are not special and are given by the above if $\binom{n-1}{n}$ is interpreted as $1/n$.

We now define biharmonic functions in the following way.

Let $$u_s = u_s^{(1)}+u_s^{(2)}+u_s^{(3)}+u_s^{(4)}, \quad s\geqslant 0,$$

where the four terms represent the contributions from the four singularities taken in the same order as above. Then

$$y_1(u_s^{(1)}+u_s^{(2)}-u_s^{(3)}-u_s^{(4)})$$

will be a biharmonic function with the right type of invariance. It is evident that if we now change the signs of all the terms in the equations (3·6), (3·7) which contain powers of $2q$ or $2l$, and if we denote by ${}^s\alpha'_n$, etc. the modified values of ${}^s\alpha_n$, etc., the new function is

$$y_1\left\{r^{-s}\cos s\theta+\sum_1^\infty r^n({}^s\alpha'_n\cos n\theta+{}^s\beta'_n\sin n\theta)\right\}.$$

A more convenient standard function is, for $s>2$,

$$\begin{aligned} \mathrm{V}_s &= v_{s-2}+2y_1\left\{r^{-s+1}\cos(s-1)\,\theta+\sum_{n=1}^\infty r^n({}^{s-1}\alpha'_n\cos n\theta+{}^{s-1}\beta'_n\sin n\theta)\right\}\\ &= r^{-s+2}\sin s\theta+\sum_{n=0}^\infty r^n\{({}^s\mathrm{C}_n+{}^s\mathrm{C}'_n r^2)\cos n\theta+({}^s\mathrm{D}_n+{}^s\mathrm{D}'_n r^2)\sin n\theta\}, \end{aligned} \quad (3\cdot8)$$

where, for $n+s$ even,

$$\left.\begin{aligned}
{}^{s}\mathrm{C}_n &= \binom{n+s-2}{n}\frac{\sin(n+s-2)\,\phi}{(2l)^{n+s-2}},\\
{}^{s}\mathrm{C}'_n &= -\binom{n+s-1}{n+1}\frac{\sin(n+s)\,\phi}{(2l)^{n+s}},\\
{}^{s}\mathrm{D}_n &= \binom{n+s-2}{n}\left\{\frac{1}{(2p)^{n+s-2}}-\frac{(-)^{\frac{1}{2}(s-n)}}{(2q)^{n+s-2}}-\frac{\cos(n+s-2)\,\phi}{(2l)^{n+s-2}}\right\},\\
{}^{s}\mathrm{D}'_n &= -\binom{n+s-1}{n+1}\left\{\frac{1}{(2p)^{n+s}}+\frac{(-)^{\frac{1}{2}(s-n)}}{(2q)^{n+s}}-\frac{\cos(n+s)\,\phi}{(2l)^{n+s}}\right\};
\end{aligned}\right\} \tag{3·9}$$

and for $n+s$ odd,

$$\left.\begin{aligned}
{}^{s}\mathrm{C}_n &= -\binom{n+s-2}{n}\left\{\frac{(-)^{\frac{1}{2}(s-n+1)}}{(2q)^{n+s-2}}+\frac{\sin(n+s-2)\,\phi}{(2l)^{n+s-2}}\right\},\\
{}^{s}\mathrm{C}'_n &= \binom{n+s-1}{n+1}\left\{\frac{(-)^{\frac{1}{2}(s-n-1)}}{(2q)^{n+s}}+\frac{\sin(n+s)\,\phi}{(2l)^{n+s}}\right\},\\
{}^{s}\mathrm{D}_n &= -\binom{n+s-2}{n}\left\{\frac{1}{(2p)^{n+s-2}}-\frac{\cos(n+s-2)\,\phi}{(2l)^{n+s-2}}\right\},\\
{}^{s}\mathrm{D}'_n &= \binom{n+s-1}{n+1}\left\{\frac{1}{(2p)^{n+s}}-\frac{\cos(n+s)\,\phi}{(2l)^{n+s}}\right\}.
\end{aligned}\right\} \tag{3·10}$$

A second standard function is, $s>2$,

$$\begin{aligned}
\mathrm{U}_s &= u_{s-2}-2y_1\left\{r^{-s+1}\sin(s-1)\,\theta+\sum_{n=1}^{\infty}r^n({}^{s-1}\gamma'_n\cos n\theta+{}^{s-1}\delta'_n\sin n\theta)\right\}\\
&= r^{-s+2}\cos s\theta+\sum_{n=0}^{\infty}r^n\{({}^{s}\mathrm{A}_n+{}^{s}\mathrm{A}'_n r^2)\cos n\theta+({}^{s}\mathrm{B}_n+{}^{s}\mathrm{B}'_n r^2)\sin n\theta\},
\end{aligned} \tag{3·11}$$

with coefficients given by

$$n+s \text{ even:}\quad \left.\begin{aligned}
{}^{s}\mathrm{A}_n &= \binom{n+s-2}{n}\left\{\frac{1}{(2p)^{n+s-2}}-\frac{(-)^{\frac{1}{2}(s-n)}}{(2q)^{n+s-2}}+\frac{\cos(n+s-2)\,\phi}{(2l)^{n+s-2}}\right\},\\
{}^{s}\mathrm{A}'_n &= -\binom{n+s-1}{n+1}\left\{\frac{1}{(2p)^{n+s}}+\frac{(-)^{\frac{1}{2}(s-n)}}{(2q)^{n+s}}+\frac{\cos(n+s)\,\phi}{(2l)^{n+s}}\right\},\\
{}^{s}\mathrm{B}_n &= \binom{n+s-2}{n}\frac{\sin(n+s-2)\,\phi}{(2l)^{n+s-2}},\\
{}^{s}\mathrm{B}'_n &= -\binom{n+s-1}{n+1}\frac{\sin(n+s)\,\phi}{(2l)^{n+s}};
\end{aligned}\right\} \tag{3·12}$$

$n+s$ odd:

$$\left.\begin{aligned} {}^sA_n &= -\binom{n+s-2}{n}\left\{\frac{1}{(2p)^{n+s-2}}+\frac{\cos(n+s-2)\,\phi}{(2l)^{n+s-2}}\right\},\\ {}^sA'_n &= \binom{n+s-1}{n+1}\left\{\frac{1}{(2p)^{n+s}}+\frac{\cos(n+s)\,\phi}{(2l)^{n+s}}\right\},\\ {}^sB_n &= -\binom{n+s-2}{n}\left\{\frac{(-)^{\frac{1}{2}(s-n-1)}}{(2q)^{n+s-2}}+\frac{\sin(n+s-2)\,\phi}{(2l)^{n+s-2}}\right\},\\ {}^sB'_n &= \binom{n+s-1}{n+1}\left\{\frac{(-)^{\frac{1}{2}(s-n+1)}}{(2q)^{n+s}}+\frac{\sin(n+s)\,\phi}{(2l)^{n+s}}\right\}.\end{aligned}\right\} \tag{3·13}$$

When $s=2$ the functions have the modified forms

$$\begin{aligned} U_2 &= 1-2y_1\left\{r^{-1}\sin\theta+\sum_{n=1}^{\infty}r^n({}^1\gamma'_n\cos n\theta+{}^1\delta'_n\sin n\theta)\right\}\\ &= \cos 2\theta+\sum_{n=0}^{\infty}r^n\{({}^2A_n+{}^2A'_nr^2)\cos n\theta+({}^2B_n+{}^2B'_nr^2)\sin n\theta\}.\\ V_2 &= 2y_1\left\{r^{-1}\cos\theta+\sum_{n=1}^{\infty}r^n({}^1\alpha'_n\cos n\theta+{}^1\beta'_n\sin n\theta)\right\}\\ &= \sin 2\theta+\sum_{n=0}^{\infty}r^n\{({}^2C_n+{}^2C'_nr^2)\cos n\theta+({}^2D_n+{}^2D'_nr^2)\sin n\theta\}.\end{aligned}$$

The coefficients ${}^2A_n$, etc. are not special and are given by the general expressions above.

As in the previous section $U_1$, $V_1$ are not required and the functions $U_0$, $V_0$ are obtainable, if wanted, from the expansions of

$$U_0=\sum_{n=1}^{4}\chi_1(z_n),\quad V_0=\sum_{n=1}^{4}\chi_2(z_n),$$

where $\chi_1(z)$ is given by (2·9) and $\chi_2(z)$ by (2·11).

## 4. An infinite double row of circles in an infinite plane

Let the circles be arranged as in fig. 3 with their centres at the points $(\pm na, 0)$, $(\pm na+p, q)$ referred to axes OXY. Let O′X′Y′ be axes with origin at O′ whose co-ordinates referred to OXY are $(p, q)$.

If
$$z=x+iy,\quad z'=x'+iy',$$
then
$$z'=p+iq-z. \tag{4·1}$$

Harmonic functions having the right sort of invariance are defined by

$$\begin{aligned} -w_0 &= \log\sin\pi z/a+\log\sin\pi z'/a\\ &= \log\sin\pi\zeta+\log\sin\pi(\zeta_0-\zeta),\end{aligned} \tag{4·2}$$

where
$$\zeta=z/a,\quad \zeta_0=(p+iq)/a;$$

and by
$$w_s = \frac{1}{(s-1)!}\frac{d^s}{d\zeta^s}\{(-)^{s-1}\log\sin\pi\zeta - \log\sin\pi(\zeta_0-\zeta)\}. \tag{4·3}$$

These functions may be expanded as follows:
$$w_0 = -\log\zeta + \sum_{n=1}^{\infty} {}^0a_n\zeta^n, \tag{4·4}$$

where
$$\left.\begin{aligned} {}^0a_n &= {}^0a_n' + {}^0a_n'', \\ {}^0a_{2n}' &= \sigma_{2n}/n, \quad {}^0a_{2n+1}' = 0, \\ {}^0a_n'' &= \pi^n f_n(c)/n!; \end{aligned}\right\} \tag{4·5}$$

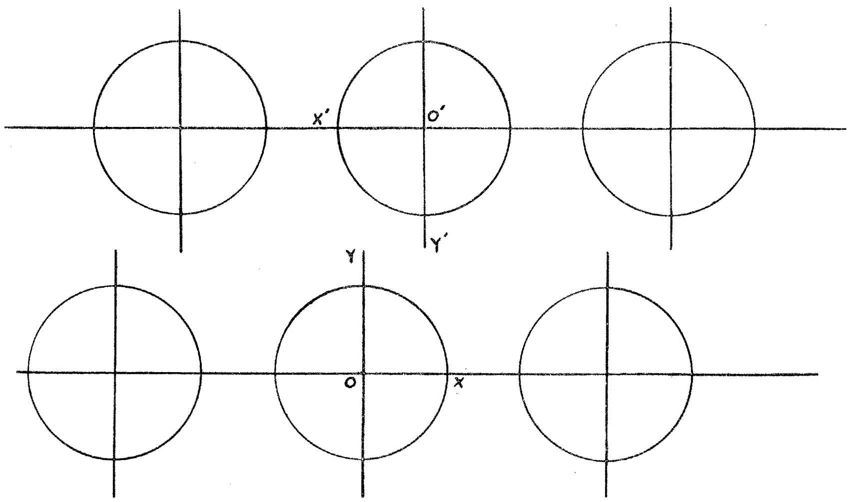


FIG. 3

with
$$\left.\begin{aligned} \sigma_n &= \sum_{\kappa=1}^{\infty}\kappa^{-n}, \\ f_1(x) &= x, \quad f_n(x) = \left\{(1+x^2)\frac{d}{dx}\right\}^{n-1}x, \quad (n>1), \\ c &= \cot\pi\zeta_0. \end{aligned}\right\} \tag{4·6}$$

The expansion for $w_s$ follows at once and is
$$w_s = \zeta^{-s} + \sum_{n=0}^{\infty} {}^sa_n\zeta^n, \tag{4·7}$$

$$\left.\begin{aligned} {}^sa_n &= {}^sa_n' + {}^sa_n'', \\ {}^{2s}a_{2n}' &= 2\binom{2n+2s-1}{2n}\sigma_{2n+2s}, \\ {}^{2s}a_{2n+1}' &= {}^{2s+1}a_{2n}' = 0, \\ {}^{2s+1}a_{2n+1}' &= -2\binom{2n+2s+1}{2n+1}\sigma_{2n+2s+2}, \\ {}^sa_n'' &= \pi^{n+s}f_{n+s}(c)/n!(s-1)!. \end{aligned}\right\} \tag{4·8}$$

The real and imaginary parts of $w_s$ provide suitable potential functions of the usual form: if

$$\rho e^{i\theta} = \zeta w_s = u_s - iv_s, \tag{4·9}$$

$$\left.\begin{aligned} u_0 &= -\log\rho + \sum_{n=0}^{\infty} \rho^n({}^0\alpha_n \cos n\theta + {}^0\beta_n \sin n\theta),\\ v_0 &= \theta + \sum_{n=0}^{\infty} \rho^n({}^0\gamma_n \cos n\theta + {}^0\delta_n \sin n\theta),\\ u_s &= \rho^{-s}\cos s\theta + \sum_{n=0}^{\infty} \rho^n({}^s\alpha_n \cos n\theta + {}^s\beta_n \sin n\theta),\\ v_s &= \rho^{-s}\sin s\theta + \sum_{n=0}^{\infty} \rho^n({}^s\gamma_n \cos n\theta + {}^s\delta_n \sin n\theta).\end{aligned}\right\} \tag{4·10}$$

Explicit forms for the coefficients are easily written down in terms of the real and imaginary parts of $f_n(c)$, but are rather complicated in character. We shall content ourselves by considering three special cases in which there is some simplification.

*The parallel position*

The first special case is that for $p = 0$ when the lattice of centres is rectangular. This will be called the parallel position.

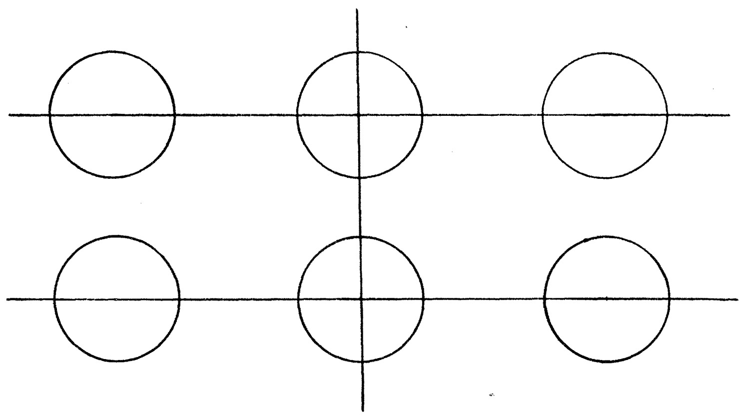

FIG. 4

We have here $$\zeta_0 = iq/a, \quad c = i\coth \pi q/a = ic'. \tag{4·11}$$

Hence $f_{2n}(c)$ is real and $f_{2n+1}(c)$ is a pure imaginary; writing

$$\pi^{2n} f_{2n}(c)/(2n)! = b_{2n}, \quad \pi^{2n+1} f_{2n+1}(c)/(2n+1)! = ib_{2n+1}, \tag{4·12}$$

we obtain

$$\left.\begin{aligned} {}^0\alpha_{2n} &= -{}^0\delta_{2n} = b_{2n} + \sigma_{2n}/n,\\ {}^0\beta_{2n+1} &= {}^0\gamma_{2n+1} = -b_{2n+1},\\ {}^0\beta_{2n} &= {}^0\gamma_{2n} = {}^0\alpha_{2n+1} = {}^0\delta_{2n+1} = 0.\end{aligned}\right\} \tag{4·13}$$

For the general function the coefficients become

$$\left.\begin{aligned}
{}^{2s}\alpha_{2n} &= -{}^{2s}\delta_{2n} = \binom{2n+2s-1}{2n}\{2\sigma_{2n+2s}+(2n+2s)\,b_{2n+2s}\},\\
{}^{2s+1}\alpha_{2n+1} &= -{}^{2s+1}\delta_{2n+1} = \binom{2n+2s+1}{2n+1}\{-2\sigma_{2n+2s+2}+(2n+2s+2)\,b_{2n+2s+2}\},\\
{}^{s}\alpha_{n} &= {}^{s}\delta_{n} = 0, \quad n+s \text{ odd},\\
{}^{s}\beta_{n} &= {}^{s}\gamma_{n} = 0, \quad n+s \text{ even},\\
{}^{s}\gamma_{n} &= {}^{s}\beta_{n} = \binom{n+s-1}{n}(n+s)\,b_{n+s}, \quad n+s \text{ odd}.
\end{aligned}\right\} \tag{4·14}$$

To obtain suitable biharmonic functions we first modify the coefficients by changing the signs of all terms arising from the second row of centres. Let the changed coefficients be denoted by ${}^{s}\alpha'_{n}$, etc., as before. The subsequent procedure is exactly as in the previous section and the equations (3·8), (3·11) hold when the new ${}^{s}\alpha'_{n}$, etc. are used. The coefficients are found to be, for $U_s$,

$$\left.\begin{aligned}
{}^{2s}A_{2n} &= \binom{2n+2s-2}{2n}\{2\sigma_{2n+2s-2}+(2n+2s-2)\,b_{2n+2s-2}\},\\
{}^{2s+1}A_{2n+1} &= \binom{2n+2s}{2n+1}\{-2\sigma_{2n+2s}+(2n+2s)\,b_{2n+2s}\},\\
{}^{2s}A'_{2n} &= -\binom{2n+2s-1}{2n+1}\{2\sigma_{2n+2s}+(2n+2s)\,b_{2n+2s}\},\\
{}^{2s+1}A'_{2n+1} &= -\binom{2n+2s+1}{2n+2}\{-2\sigma_{2n+2s+2}+(2n+2s+2)\,b_{2n+2s+2}\},\\
{}^{s}A_{n} &= {}^{s}A'_{n} = 0, \quad n+s \text{ odd},
\end{aligned}\right\} \tag{4·15}$$

$$\left.\begin{aligned}
{}^{s}B_{n} &= {}^{s}B'_{n} = 0, \quad n+s \text{ even},\\
{}^{s}B_{n} &= \binom{n+s-2}{n}(n+s-2)\,b_{n+s-2}, \quad n+s \text{ odd},\\
{}^{s}B'_{n} &= -\binom{n+s-1}{n+1}(n+s)\,b_{n+s}, \quad n+s \text{ odd}.
\end{aligned}\right\} \tag{4·16}$$

For $V_s$ we have, for all values of $n$ and $s$,

$$\left.\begin{aligned}
{}^{s}C_{n} &= {}^{s}B_{n}, & {}^{s}C'_{n} &= {}^{s}B'_{n},\\
{}^{s}D_{n} &= -{}^{s}A_{n}, & {}^{s}D'_{n} &= -{}^{s}A'_{n}.
\end{aligned}\right\} \tag{4·17}$$

The functions $U_2$, $V_2$ are not special, while $U_0$, $V_0$ may be obtained in a manner similar to that used previously. They are not of particular interest here, for in most problems with these boundaries $u_0$ may be used instead of $U_0$, etc.

*The alternate position*

The next special case in which there is some simplification is that in which $p = \frac{1}{2}a$. The circles are arranged as shown in fig. 5.

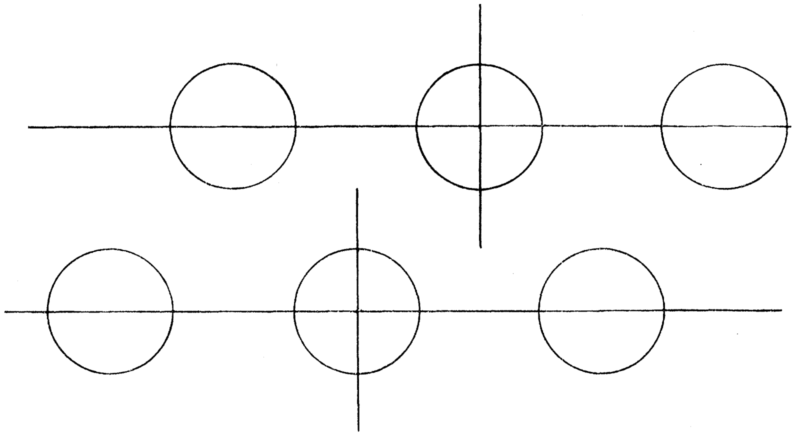

FIG. 5

The particular value of the quantity $c$ is now

$$c = \cot(i\pi q/a + \pi/2) = i\tan(q/a) = it. \tag{4·18}$$

Let $$\pi^{2n} f_{2n}(c)/(2n)! = b'_{2n}, \quad \pi^{2n+1} f_{2n+1}(c)/(2n+1)! = ib'_{2n+1}. \tag{4·19}$$

Consequently the harmonic and biharmonic functions for this set of boundaries are obtained from the equations (4·13)–(4·17) when $b_n$ has been replaced by $b'_n$.

*A row of pairs of circles*

A third special case is that for $q = 0$. The boundaries are now pairs of circles with their centres collinear (fig. 6).

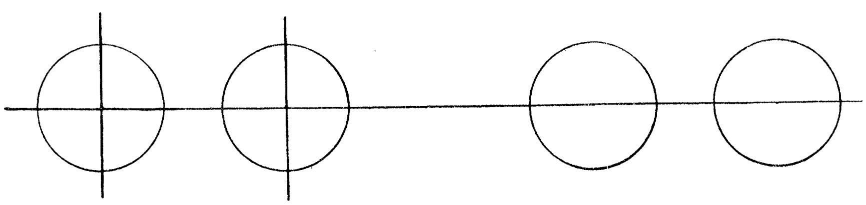

FIG. 6

The coefficients in the harmonic functions simplify at once and are

$${}^s\alpha_n = {}^s a_n, \quad {}^s\beta_n = 0, \quad {}^s\gamma_n = 0, \quad {}^s\delta_n = -{}^s a_n. \tag{4·20}$$

Since now $$c = \cot(\pi p/a), \qquad \text{let} \quad \pi^n f_n(c)/n! = d_n. \tag{4·21}$$



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The biharmonic functions have the same form as before with the special coefficients given by

$$\left.\begin{aligned} {}^{2s}\mathrm{A}_{2n} &= \binom{2n+2s-2}{2n}\{2\sigma_{2n+2s-2}+(2n+2s-2)\,d_{2n+2s-2}\}, \\ {}^{2s+1}\mathrm{A}_{2n+1} &= \binom{2n+2s-1}{2n+1}\{-2\sigma_{2n+2s}+(2n+2s)\,d_{2n+2s}\}, \\ {}^{2s}\mathrm{A}'_{2n} &= -\binom{2n+2s-1}{2n+1}\{2\sigma_{2n+2s}+(2n+2s)\,d_{2n+2s}\}, \\ {}^{2s+1}\mathrm{A}_{2n+1} &= -\binom{2n+2s+1}{2n+2}\{-2\sigma_{2n+2s+2}+(2n+2s+2)\,d_{2n+2s+2}\}. \end{aligned}\right\} \tag{4·22}$$

$$\left.\begin{aligned} {}^{s}\mathrm{A}_{n} &= {}^{s}\mathrm{A}'_{n} = 0, \quad n+s \text{ odd}, \\ {}^{s}\mathrm{B}_{n} &= {}^{s}\mathrm{B}'_{n} = {}^{s}\mathrm{C}_{n} = {}^{s}\mathrm{C}'_{n} = 0, \\ {}^{s}\mathrm{D}_{n} &= -{}^{s}\mathrm{A}_{n}, \quad {}^{s}\mathrm{D}'_{n} = {}^{s}\mathrm{A}'_{n}. \end{aligned}\right\} \tag{4·23}$$

## 5. The expansions for forces in a strip

For the problem of circular holes in a strip we take as our fundamental singularity a force acting at a point which is to be the centre of the hole. Moreover, this force must be given by a stress function which produces zero stresses on the parallel straight boundaries. We have, therefore, to consider the expansion of such a function about its centre and about other points in the strip. When this has been done suitable singularities will be taken and combined to give the required functions in the various cases.

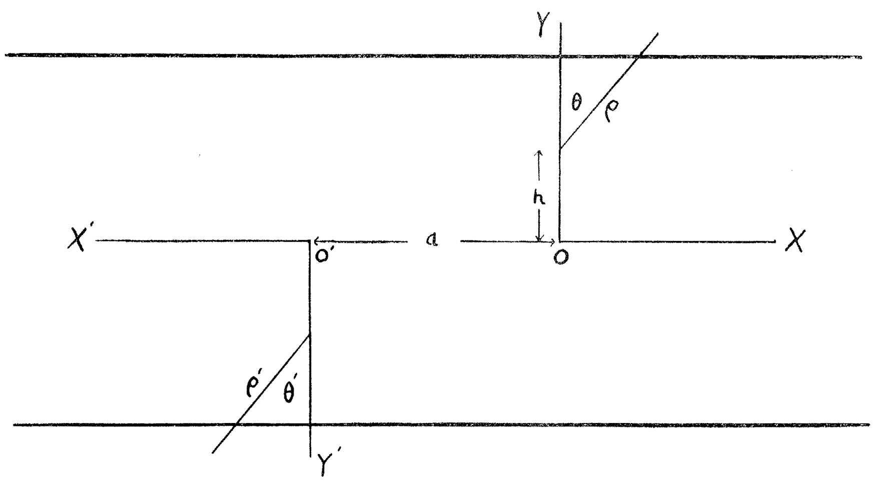


Fig. 7

*(a) Longitudinal force acting at (o, h) expanded about (o, h)*

Let the strip be bounded in the $xy$ plane by the lines $y = \pm b$. Polar co-ordinates are chosen so that

$$x = r\sin\theta, \quad y-h = r\cos\theta. \tag{5·1}$$

The force P, acting at $(o, h)$, is taken in direction $\overrightarrow{OX}$; then writing

$$x/b = \xi, \quad y/b = \eta, \quad h/b = \alpha, \tag{5·2}$$

the stress function is (see Howland 1929)

$$\chi = \frac{Pb}{4\pi(1-\sigma)} \{(1-2\sigma)\,\xi \log \rho + 2(1-\sigma)\,(\eta-\alpha)\,\theta + \Phi\}, \tag{5·3}$$

where
$$\Phi = \chi_1 + \chi_2 + \chi_3 + \chi_4,$$

and

$$\left.\begin{aligned}
\chi_1 &= \int_0^\infty \frac{u\eta s S - (s+uc)\,C}{u^2\Sigma}\,(B_1+B_2)\sin u\xi\,du,\\
\chi_2 &= \int_0^\infty \frac{sC-\eta cS}{u\Sigma}\,(B_1'+B_2')\sin u\xi\,du,\\
\chi_3 &= \int_0^\infty \frac{u\eta cC-(c+us)\,S}{u^2\Sigma'}\,(B_1-B_2)\sin u\xi\,du,\\
\chi_4 &= \int_0^\infty \frac{cS-\eta sC}{u\Sigma'}\,(B_1'-B_2')\sin u\xi\,du.
\end{aligned}\right\} \tag{5·4}$$

The new symbols denote

$$\left.\begin{aligned}
&s = \sinh u, \quad c = \cosh u, \quad S = \sinh u\eta, \quad C = \cosh u\eta,\\
&\Sigma = \sinh 2u + 2u, \quad \Sigma' = \sinh 2u - 2u;
\end{aligned}\right\} \tag{5·5}$$

and

$$\left.\begin{aligned}
B_1 &= [u(1-\alpha)-(1-2\sigma)]\,e^{-u(1-\alpha)},\\
B_2 &= [u(1+\alpha)-(1-2\sigma)]\,e^{-u(1+\alpha)},\\
B_1' &= [2(1-\sigma)-u(1-\alpha)]\,e^{-u(1-\alpha)},\\
B_2' &= [2(1-\sigma)-u(1+\alpha)]\,e^{-u(1+\alpha)}.
\end{aligned}\right\} \tag{5·6}$$

To obtain the expansion about the singularity, i.e. the point $(o, h)$, write

$$\left.\begin{aligned}
&\xi = \rho\sin\theta, \quad \eta-\alpha = \eta' = \rho\cos\theta,\\
&C' = \cosh u\eta', \quad S' = \sinh u\eta',\\
&c_2 = \cosh 2\alpha u, \quad s_2 = \sinh 2\alpha u.
\end{aligned}\right\} \tag{5·7}$$

Then substituting the values of $B_1$, etc. into equations (5·4), we get, after some reduction,

$$\left.\begin{aligned}
\chi_1+\chi_2 &= \int_0^\infty \frac{\sin u\xi}{u^2\Sigma}\{U_1C'+U_2S'+U_3\eta'C'+U_4\eta'S'\}\,du,\\
\chi_3+\chi_4 &= \int_0^\infty \frac{\sin u\xi}{u^2\Sigma}\{V_1C'+V_2S'+V_3\eta'C'+V_4\eta'S'\}\,du,
\end{aligned}\right\} \tag{5·8}$$

where

$$\mathrm{U}_1 = 2\alpha u^2 s_2 - (1+\alpha^2)\, u^2 c_2 - (1-\alpha^2)\, u^2 - \alpha u s_2 e^{-2u} + \tfrac{1}{2}(1-2\sigma)\,[(c_2+1)\,(2u+1-e^{-2u}) - 2\alpha u s_2],$$

$$\mathrm{U}_2 = \alpha u[(2u - e^{-2u})\, c_2 - 1] - (1+\alpha^2)\, u^2 s_2 - (1-2\sigma)\,[(c_2+1)\,\alpha u - \tfrac{1}{2}(2u+1-e^{-2u})\, s_2],$$

$$\mathrm{U}_3 = \tfrac{1}{2}(2u-1-e^{-2u})\, u s_2 - \alpha u^2 (c_2-1) - (1-2\sigma)\, u s_2,$$

$$\mathrm{U}_4 = \tfrac{1}{2}(2u-1-e^{-2u})\,(c_2+1)\, u - \alpha u^2 s_2 - (1-2\sigma)\,(c_2+1)\, u,$$

$$\mathrm{V}_1 = (2u+e^{-2u})\,\alpha u s_2 - (1+\alpha^2)\, u^2 c_2 + (1-\alpha^2)\, u^2 - (1-2\sigma)\,[\alpha u s_2 - \tfrac{1}{2}(2u+1+e^{-2u})\,(c_2-1)],$$

$$\mathrm{V}_2 = (2u+e^{-2u})\,\alpha u c_2 - (1+\alpha^2)\, u^2 s_2 + \alpha u - (1-2\sigma)\,[\alpha u (c_2-1) - \tfrac{1}{2}(2u+1+e^{-2u})\, s_2],$$

$$\mathrm{V}_3 = \tfrac{1}{2}(2u-1+e^{-2u})\, u s_2 - \alpha u^2 (c_2+1) - (1-2\sigma)\, u s_2,$$

$$\mathrm{V}_4 = \tfrac{1}{2}(2u-1+e^{-2u})\, u (c_2-1) - \alpha u^2 s_2 - (1-2\sigma)\, u (c_2-1).$$

In the integrals of (5·8) we now use the expansions

$$\left.\begin{aligned}
\mathrm{C}' \sin u\xi &= \sum_{n=0}^{\infty} \frac{(u\rho)^{2n+1}}{(2n+1)!} \sin (2n+1)\,\theta, \\
\mathrm{S}' \sin u\xi &= \sum_{n=1}^{\infty} \frac{(u\rho)^{2n}}{(2n)!} \sin 2n\theta, \\
\eta' \mathrm{C}' \sin u\xi &= \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \frac{u^{2n+1}\rho^{2n+2}}{(2n+1)!} + \frac{u^{2n-1}\rho^{2n}}{(2n-1)!} \right\} \sin 2n\theta, \\
\eta' \mathrm{S}' \sin u\xi &= \frac{1}{2} \left[ \frac{u^2 \rho^3 \sin\theta}{2} + \sum_{n=1}^{\infty} \left\{ \frac{u^{2n}\rho^{2n+1}}{(2n)!} + \frac{u^{2n+1}\rho^{2n+3}}{(2n+2)!} \right\} \sin (2n+1)\,\theta \right].
\end{aligned}\right\} \tag{5·9}$$

Let

$$\left.\begin{aligned}
&\int_0^\infty \frac{u^n}{\Sigma}\, du = \mathrm{I}_n, \qquad \int_0^\infty \frac{u^n e^{-2u}}{\Sigma}\, du = \mathrm{J}_n, \\
&\int_0^\infty \frac{u^n \cosh 2\alpha u}{\Sigma}\, du = \mathrm{C}_n, \qquad \int_0^\infty \frac{u^n \cosh 2\alpha u e^{-2u}}{\Sigma}\, du = \mathrm{D}_n, \\
&\int_0^\infty \frac{u^n \sinh 2\alpha u}{\Sigma}\, du = \mathrm{S}_n, \qquad \int_0^\infty \frac{u^n \sinh 2\alpha u e^{-2u}}{\Sigma}\, du = \mathrm{T}_n,
\end{aligned}\right\} \tag{5·10}$$

and denote with primes the corresponding integrals in which the $\Sigma$ is replaced by $\Sigma'$.

Then provided the order of integration and summation may be changed we obtain the series

$$\left.\begin{aligned}
\chi_1 + \chi_2 &= \sum_{n=1}^{\infty} (\mathrm{A}'_n + \mathrm{B}'_n \rho^2)\, \rho^n \sin n\theta, \\
\chi_3 + \chi_4 &= \sum_{n=1}^{\infty} (\mathrm{A}''_n + \mathrm{B}''_n \rho^2)\, \rho^n \sin n\theta,
\end{aligned}\right\} \tag{5·11}$$



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where

$$\left.\begin{aligned}
A'_{2n} &= \frac{1}{(2n)!}\{(n-1)\,\alpha I_{2n-1}+2\alpha C_{2n}-n\alpha C_{2n-1}-(1+\alpha^2)\,S_{2n}+nS_{2n-1}-\tfrac{1}{2}nS_{2n-2}\\
&\quad -\alpha D_{2n-1}-\tfrac{1}{2}nT_{2n-2}-(1-2\sigma)\,[\alpha(C_{2n-1}+I_{2n-1})-S_{2n-1}\\
&\quad +(n-\tfrac{1}{2})\,S_{2n-2}+\tfrac{1}{2}T_{2n-2}]\},\\
A'_{2n+1} &= \frac{1}{(2n+1)!}\{-(1-\alpha^2)\,I_{2n+1}-(1+\alpha^2)\,C_{2n+1}+2\alpha S_{2n+1}-\alpha T_{2n}\}\\
&\quad +\frac{1}{2(2n)!}\{(I_{2n}+C_{2n})-\tfrac{1}{2}(I_{2n-1}+C_{2n-1})-\alpha S_{2n}-\tfrac{1}{2}(D_{2n-1}+J_{2n-1})\}\\
&\quad +\frac{(1-2\sigma)}{(2n+1)!}\{(I_{2n}+C_{2n})-n(I_{2n-1}+C_{2n-1})-\alpha S_{2n}-\tfrac{1}{2}(D_{2n-1}+J_{2n-1}),\\
B'_{2n} &= \frac{1}{2(2n+1)!}\{\alpha(I_{2n+1}-C_{2n+1})+S_{2n+1}-\tfrac{1}{2}S_{2n}-\tfrac{1}{2}T_{2n}-(1-2\sigma)\,S_{2n}\},\\
B'_{2n+1} &= \frac{1}{2(2n+2)!}\{(I_{2n+2}+C_{2n+2})-\tfrac{1}{2}(I_{2n+1}+C_{2n+1})-\alpha S_{2n+2}\\
&\quad -\tfrac{1}{2}(J_{2n+1}+D_{2n+1})-(1-2\sigma)\,(I_{2n+1}+C_{2n+1})\},
\end{aligned}\right\} \quad (5\cdot 12)$$

$$\left.\begin{aligned}
A''_{2n} &= \frac{1}{(2n)!}\{-(n-1)\,\alpha I'_{2n-1}+2\alpha C'_{2n}-n\alpha C'_{2n-1}-(1+\alpha^2)\,S'_{2n}+nS'_{2n-1}-\tfrac{1}{2}nS'_{2n-2}\\
&\quad +\alpha D'_{2n-1}+\tfrac{1}{2}nT'_{2n-2}-(1-2\sigma)\,[\alpha(C'_{2n-1}-I'_{2n-1})-S'_{2n-1}\\
&\quad +(n-\tfrac{1}{2})\,S'_{2n-2}-\tfrac{1}{2}T'_{2n-2}]\},\\
A''_{2n+1} &= \frac{1}{(2n+1)!}\{(1-\alpha^2)\,I'_{2n+1}-(1+\alpha^2)\,C'_{2n+1}+2\alpha S'_{2n+1}+\alpha T'_{2n}\}\\
&\quad +\frac{1}{2(2n)!}\{(C'_{2n}-I'_{2n})-\tfrac{1}{2}(C'_{2n-1}-I'_{2n-1})-\alpha S'_{2n}+\tfrac{1}{2}(D'_{2n-1}-J'_{2n-1})\}\\
&\quad -\frac{(1-2\sigma)}{(2n+1)!}\{-(C'_{2n}-I'_{2n})+n(C'_{2n-1}-I'_{2n-1})+\alpha S'_{2n}-\tfrac{1}{2}(D'_{2n-1}-J'_{2n-1})\},\\
B''_{2n} &= \frac{1}{2(2n+1)!}\{-\alpha(C'_{2n+1}+I'_{2n+1})+S'_{2n+1}-\tfrac{1}{2}S'_{2n}+\tfrac{1}{2}T'_{2n}-(1-2\sigma)\,S'_{2n}\},\\
B''_{2n+1} &= \frac{1}{2(2n+2)!}\{(C'_{2n+2}-I'_{2n+2})-\tfrac{1}{2}(C'_{2n+1}-I'_{2n+1})-\alpha S'_{2n+2}\\
&\quad +\tfrac{1}{2}(D'_{2n+1}-J'_{2n+1})-(1-2\sigma)\,(C'_{2n+1}-I'_{2n+1})\}.
\end{aligned}\right\} \quad (5\cdot 13)$$

We thus have the stress function given by

$$\chi = \frac{Pb}{4\pi(1-\sigma)}\Big\{(1-2\sigma)\,\rho\sin\theta\log\rho+2(1-\sigma)\,\theta\rho\cos\theta + \sum_{n=1}^{\infty}[(A'_n+A''_n)+(B'_n+B''_n)\rho^2]\,\rho^n\sin n\theta\Big\}. \quad (5\cdot 14)$$

It will be noted that some of the integrals in the above coefficients are divergent at the lower limit. The convergence of the integrals (5·8) has been discussed (see Howland and Stevenson 1933), but the coefficients we shall use require some investigation. The divergent integrals occur when $n$ is zero or unity in the coefficients $A_1'$, $A_1''$, $A_2''$, $B_1''$. The first and second of these, since they multiply terms producing zero stresses, may be put equal to zero. Consider $A_2''$, this is

$$\tfrac{1}{2}\{2\alpha C_2' - \alpha C_1' + \alpha D_1' - (1+\alpha^2)\, S_2' + S_1' - \tfrac{1}{2}S_0' + \tfrac{1}{2}T_0' \\ - (1-2\sigma)\,[\alpha(C_1' - I_1') - S_1' + \tfrac{1}{2}S_0' - \tfrac{1}{2}T_0']\}.$$

Of these $S_2'$ is convergent at the lower limit but the others diverge. If, however, we take the combination $2C_2' - C_1' + D_1'$, we may interpret it as the single convergent integral

$$\int_0^\infty \frac{u(e^{-2u} - 1 + 2u)\cosh 2\alpha u}{\Sigma'}\, du.$$

Similarly, the other combinations are interpreted as

$$2S_1' - S_0' + T_0' = \int_0^\infty \frac{(2u - 1 + e^{-2u})\sinh 2\alpha u}{\Sigma'}\, du,$$

and
$$C_1' - I_1' = \int_0^\infty \frac{(\cosh 2\alpha u - 1)\, u}{\Sigma'}\, du,$$

which converge.

In the other coefficients one other combination occurs which has to be considered. This is

$$D_1' - J_1' = \int_0^\infty \frac{(\cosh 2\alpha u - 1)\, u e^{-2u}}{\Sigma'}\, du,$$

which presents no difficulty.

### (*b*) *Longitudinal force acting at* $(-a, -h)$ *expanded about* $(o, h)$

A force P acting at $(-a, -h)$ in direction $\overrightarrow{OX}$ is given by a stress function

$$\chi' = -\frac{Pb}{4\pi(1-\sigma)}\{(1-2\sigma)\,\xi' \log e' + 2(1-\sigma)\,(\eta' - \alpha')\,\theta' + \Phi'\}, \tag{5·15}$$

the relation between the accented co-ordinates and the unaccented being

$$\left.\begin{aligned} \xi' &= x'/b = -(x+a)/b = -(\xi+\beta), \\ \eta' &= y'/b = -y/b = -\eta, \\ \rho' &= [(\xi+\beta)^2 + (\eta+\alpha)^2]^{\frac{1}{2}}, \quad \tan\theta' = (\xi+\beta)/(\eta+\alpha). \end{aligned}\right\} \tag{5·16}$$

Expansions of $\xi' \log \rho'$ and $(\eta' - \alpha)\,\theta'$ are first required. These may be obtained as follows: let

$$\zeta = \xi + i(\eta - \alpha) = i\rho e^{-i\theta},$$

$$\gamma = \beta + 2i\alpha = i\kappa e^{-i\phi},$$

so that $$\kappa = \sqrt{(4\alpha^2+\beta^2)}, \quad \phi = \arc\tan(\beta/2\alpha). \tag{5·17}$$

Then since $$\log(\zeta+\gamma) = \log\rho' + i(\tfrac{1}{2}\pi-\theta'),$$

and $$\log(\zeta+\gamma) = \log\gamma + \log(1+\zeta/\gamma)$$
$$= \log\gamma + \sum_{n=1}^{\infty}\frac{(-)^{n-1}}{n}\left(\frac{\zeta}{\gamma}\right)^n,$$

we have $$\log\rho' = \log\kappa + \sum_{n=1}^{\infty}\frac{(-)^{n-1}}{n}\left(\frac{\rho}{\kappa}\right)^n\cos n(\theta-\phi),$$
$$\theta' = \phi + \sum_{n=1}^{\infty}\frac{(-)^{n-1}}{n}\left(\frac{\rho}{\kappa}\right)^n\sin n(\theta-\phi).$$

If trivial terms are omitted we may write

$$(\xi+\beta)\log\rho' = (\rho\sin\theta+\beta)\sum_{n=1}^{\infty}\frac{(-)^n}{n}\left(\frac{\rho}{\kappa}\right)^n\cos n(\theta-\phi)$$
$$= \sum_{n=0}^{\infty}\rho^n\{(l_n+m_n\rho^2)\cos n\theta + (n_n+p_n\rho^2)\sin n\theta\},$$

where $l_0$, $l_1$, $n_0$, $n_1$, $p_0$, may be taken as zero while the other coefficients are

$$\left.\begin{aligned} l_n &= (-)^{n+1}\{2n\alpha\sin n\phi+(n-2)\beta\cos n\phi\}/\kappa^n\,2n(n-1),\\ m_n &= (-)^n\{\sin(n+1)\phi\}/\kappa^{n+1}\,2(n+1),\\ n_n &= (-)^n\{2n\alpha\cos n\phi-(n-2)\beta\sin n\phi\}/\kappa^n\,2n(n-1),\\ p_n &= (-)^{n+1}\{\cos(n+1)\phi\}/\kappa^{n+1}\,2(n+1).\end{aligned}\right\} \tag{5·18}$$

Similarly, $$(\eta+\alpha)\theta' = \sum_{n=1}^{\infty}\rho^n\{(l'_n+m'_n\rho^2)\cos n\theta+(n'_n+p'_n\rho^2)\sin n\theta\},$$

where, apart from the zero coefficients as before,

$$\left.\begin{aligned} l'_n &= (-)^{n+1}\{2(n-2)\alpha\sin n\phi+n\beta\cos n\phi\}/\kappa^n\,2n(n-1),\\ m'_n &= m_n,\\ n'_n &= (-)^n\{2(n-2)\alpha\cos n\phi-n\beta\sin n\phi\}/\kappa^n\,2n(n-1),\\ p'_n &= p_n.\end{aligned}\right\} \tag{5·19}$$

The function $\Phi'$ is defined by

$$-\Phi' = \chi'_1+\chi'_2+\chi'_3+\chi'_4,$$

in which $\chi'_1$ is given by equation (5·4) for $\chi_1$ if $\xi+\beta$ be substituted for $\xi$, and instead of the values for $B_1$, etc. given by equations (5·6) we have

$$B_1 = [u(1+\alpha)-(1-2\sigma)]\,e^{-u(1+\alpha)},$$

etc., the sign of $\alpha$ being changed throughout. $\chi_2'$, $\chi_3'$, $\chi_4'$ are obtained in a similar manner. This leads to the expressions:

$$\left.\begin{aligned} \chi_1'+\chi_2' &= \int_0^\infty \frac{\sin u(\xi+\beta)}{u^2\Sigma}\{U_1C'+U_2S'+U_3\eta'C'+U_4\eta'S'\}\,du,\\ \chi_3'+\chi_4' &= -\int_0^\infty \frac{\sin u(\xi+\beta)}{u^2\Sigma'}\{V_1C'+V_2S'+V_3\eta'C'+V_4\eta'S'\}\,du. \end{aligned}\right\} \quad (5\cdot20)$$

The symbols $U_1$, etc. are those defined in the previous section. Substitution of the expansions (5·9) together with

$$\left.\begin{aligned} C'\cos u\xi &= \sum_{n=0}^{\infty}\frac{(u\rho)^{2n}}{(2n)!}\cos 2n\theta,\\ S'\cos u\xi &= \sum_{n=0}^{\infty}\frac{(u\rho)^{2n+1}}{(2n+1)!}\cos(2n+1)\theta,\\ \eta'C'\cos u\xi &= \tfrac{1}{2}\left[2\rho\cos\theta+\tfrac{1}{2}u^2\rho^3\cos\theta+\sum_{n=1}^{\infty}\left\{\frac{u^{2n}\rho^{2n+1}}{(2n)!}+\frac{u^{2n+2}\rho^{2n+3}}{(2n+2)!}\right\}\cos(2n+1)\theta\right],\\ \eta'S'\cos u\xi &= \tfrac{1}{2}\left[u\rho^2+\sum_{n=1}^{\infty}\left\{\frac{u^{2n-1}\rho^{2n}}{(2n+1)!}+\frac{u^{2n+1}\rho^{2n+2}}{(2n-1)!}\right\}\cos 2n\theta\right], \end{aligned}\right\} \quad (5\cdot21)$$

we obtain expansions of the form

$$\chi_1'+\chi_2' = \sum_{n=0}^{\infty}\rho^n\{(L_n'+M_n'\rho^2)\cos n\theta+(N_n'+P_n'\rho^2)\sin n\theta\},$$

$$\chi_3'+\chi_4' = \sum_{n=0}^{\infty}\rho^n\{(L_n''+M_n''\rho^2)\cos n\theta+(N_n''+P_n''\rho^2)\sin n\theta\}.$$

To express these new coefficients we need the following new symbols:

$$\left.\begin{aligned} &\int_0^\infty \frac{u^n}{\Sigma}\cos\beta u\,du = I_n, \quad \int_0^\infty \frac{u^n}{\Sigma}\sin\beta u\,du = \bar{I}_n,\\ &\int_0^\infty \frac{u^n e^{-2u}}{\Sigma}\cos\beta u\,du = J_n, \quad \int_0^\infty \frac{u^n e^{-2u}}{\Sigma}\sin\beta u\,du = \bar{J}_n, \end{aligned}\right\} \quad (5\cdot22)$$

and so on, the integrals (5·10) with their integrands multiplied by $\cos\beta u$ being denoted by *italics*, those multiplied by $\sin\beta u$, by *italics* with bar. Primes as before will be used to denote the corresponding integrals in which $\Sigma$ is replaced by $\Sigma'$.

Then

$$\begin{aligned} L_{2n}' = \frac{1}{(2n)!}\{&2\alpha\bar{S}_{2n}-(1+\alpha^2)\,\bar{C}_{2n}-(1-\alpha^2)\,\bar{I}_{2n}-\alpha\bar{T}_{2n-1}-n\alpha\bar{S}_{2n-1}\\ &+n(\bar{C}_{2n-1}+\bar{I}_{2n-1})-\tfrac{1}{2}n(\bar{C}_{2n-2}+\bar{I}_{2n-2}+\bar{D}_{2n-2}+\bar{J}_{2n-2})\\ &+(1-2\sigma)\,[(\bar{C}_{2n-1}+\bar{I}_{2n-1})-\tfrac{1}{2}(\bar{D}_{2n-2}+\bar{J}_{2n-2})\\ &\qquad-(n-\tfrac{1}{2})\,(\bar{C}_{2n-2}+\bar{I}_{2n-2})-\alpha\bar{S}_{2n-1}]\}, \end{aligned}$$

$$\left.\begin{aligned}
\mathrm{L}'_{2n+1} &= \frac{1}{(2n+1)!}\{2\alpha\overline{C}_{2n+1}-\alpha\overline{D}_{2n}-\alpha\overline{I}_{2n}-(1+\alpha^2)\,\overline{S}_{2n+1}\} \\
&\quad +\frac{1}{2(2n)!}\{\overline{S}_{2n}-\tfrac{1}{2}(\overline{S}_{2n-1}+\overline{T}_{2n-1})-\alpha(\overline{C}_{2n}-\overline{I}_{2n})\} \\
&\quad -\frac{(1-2\sigma)}{(2n+1)!}\{\alpha(\overline{C}_{2n}+\overline{I}_{2n})-\overline{S}_{2n}+\tfrac{1}{2}\overline{T}_{2n-1}+n\overline{S}_{2n-1}\}, \\
\mathrm{M}'_{2n} &= \frac{1}{2(2n+1)!}\{\overline{C}_{2n+1}+\overline{I}_{2n+1}-\tfrac{1}{2}(\overline{C}_{2n}+\overline{I}_{2n}+\overline{D}_{2n}+\overline{J}_{2n})-\alpha\overline{S}_{2n+1} \\
&\quad -(1-2\sigma)\,(\overline{C}_{2n}+\overline{I}_{2n})\}, \\
\mathrm{M}'_{2n+1} &= \frac{1}{2(2n+2)!}\{\overline{S}_{2n+2}-\tfrac{1}{2}(\overline{S}_{2n+1}+\overline{T}_{2n+1})-\alpha(\overline{C}_{2n+2}-\overline{I}_{2n+2})-(1-2\sigma)\overline{S}_{2n+1}\};
\end{aligned}\right\} \quad (5{\cdot}23)$$

$$\left.\begin{aligned}
\mathrm{N}'_{2n} &= \frac{1}{(2n)!}\{2\alpha C_{2n}-n\alpha C_{2n-1}-\alpha D_{2n-1}-(1+\alpha^2)\,S_{2n}+nS_{2n-1} \\
&\quad -\tfrac{1}{2}n(S_{2n-2}+T_{2n-2})+(n-1)\,\alpha I_{2n-1} \\
&\quad -(1-2\sigma)\,[\alpha C_{2n-1}+(n-\tfrac{1}{2})\,S_{2n-2}-S_{2n-1}+\tfrac{1}{2}T_{2n-2}+\alpha I_{2n-1}]\}, \\
\mathrm{N}'_{2n+1} &= \frac{1}{(2n+1)!}\{2\alpha S_{2n+1}-(1+\alpha^2)\,C_{2n+1}-\alpha T_{2n}-(1-\alpha^2)\,I_{2n+1}\} \\
&\quad +\frac{1}{2(2n)!}\{C_{2n}+I_{2n}-\tfrac{1}{2}(C_{2n-1}+D_{2n-1}+I_{2n-1}+J_{2n-1})-\alpha S_{2n}\} \\
&\quad +\frac{(1-2\sigma)}{(2n+1)!}\{C_{2n}+I_{2n}-n(C_{2n-1}+I_{2n-1})-\tfrac{1}{2}(D_{2n-1}+J_{2n-1})-\alpha S_{2n}\}, \\
\mathrm{P}'_{2n} &= \frac{1}{2(2n+1)!}\{S_{2n+1}-\tfrac{1}{2}(S_{2n}+T_{2n})-\alpha(C_{2n+1}-I_{2n+1})-(1-2\sigma)\,S_{2n}\}, \\
\mathrm{P}'_{2n+1} &= \frac{1}{2(2n+2)!}\{C_{2n+2}+I_{2n+2}-\tfrac{1}{2}(C_{2n+1}+D_{2n+1}+I_{2n+1}+J_{2n+1}) \\
&\quad -\alpha S_{2n+2}-(1-2\sigma)\,(C_{2n+1}+I_{2n+1})\},
\end{aligned}\right\} \quad (5{\cdot}24)$$

$$\left.\begin{aligned}
-\mathrm{L}''_{2n} &= \frac{1}{(2n)!}\{2\alpha\overline{S}'_{2n}+\alpha\,\overline{T}'_{2n-1}-(1+\alpha^2)\,\overline{C}'_{2n}+(1-\alpha^2)\,\overline{I}'_{2n}+n(\overline{C}'_{2n-1}-\overline{I}'_{2n-1}) \\
&\quad -\tfrac{1}{2}n(\overline{C}'_{2n-2}-\overline{D}'_{2n-2}-\overline{I}'_{2n-2}+\overline{J}'_{2n-2})-n\alpha\overline{S}'_{2n-1} \\
&\quad -(1-2\sigma)\,[\alpha\overline{S}'_{2n-1}-\overline{C}'_{2n-1}+\overline{I}'_{2n-1} \\
&\qquad +(n-\tfrac{1}{2})\,(\overline{C}'_{2n-2}-\overline{I}'_{2n-2})-\tfrac{1}{2}(\overline{D}'_{2n-2}-\overline{J}'_{2n-2})]\}, \\
-\mathrm{L}''_{2n+1} &= \frac{1}{(2n+1)!}\{2\alpha\overline{C}'_{2n+1}+\alpha\overline{D}'_{2n}-(1+\alpha^2)\,\overline{S}'_{2n+1}+\alpha\overline{I}'_{2n}\} \\
&\quad +\frac{1}{2(2n)!}\{\overline{S}'_{2n}-\tfrac{1}{2}(\overline{S}'_{2n-1}-\overline{T}'_{2n-1})-\alpha(\overline{C}'_{2n}+\overline{I}'_{2n})\} \\
&\quad -\frac{(1-2\sigma)}{(2n+1)!}\{\alpha(\overline{C}'_{2n}-\overline{I}'_{2n})-\overline{S}'_{2n}+n\overline{S}'_{2n-1}-\tfrac{1}{2}\overline{T}'_{2n-1}\}, \\
-\mathrm{M}''_{2n} &= \frac{1}{2(2n+1)!}\{\overline{C}'_{2n+1}-\overline{I}'_{2n+1}-\tfrac{1}{2}(\overline{C}'_{2n}-\overline{D}'_{2n}-\overline{I}'_{2n}+\overline{J}'_{2n})-\alpha\overline{S}'_{2n+1} \\
&\quad -(1-2\sigma)\,(\overline{C}'_{2n}-\overline{I}'_{2n})\}, \\
-\mathrm{M}''_{2n+1} &= \frac{1}{2(2n+2)!}\{\overline{S}'_{2n+2}-\tfrac{1}{2}(\overline{S}'_{2n+1}-\overline{T}'_{2n+1})-\alpha(\overline{C}'_{2n+2}+\overline{I}'_{2n+2})-(1-2\sigma)\,\overline{S}'_{2n+1}\};
\end{aligned}\right\} \quad (5{\cdot}25)$$

$$\left.\begin{aligned}
-\mathrm{N}''_{2n} &= \frac{1}{(2n)!}\{2\alpha C'_{2n} - n\alpha C'_{2n-1} - (n-1)\,\alpha I'_{2n-1} + \alpha D'_{2n-1} - (1+\alpha^2)\,S'_{2n} \\
&\qquad + nS'_{2n-1} - \tfrac{1}{2}n(S'_{2n-2} + T'_{2n-2}) \\
&\qquad - (1-2\sigma)\,[\alpha(C'_{2n-1} - I'_{2n-1}) - S'_{2n-1} + (n-\tfrac{1}{2})\,S'_{2n-2} - \tfrac{1}{2}T'_{2n-2}]\}, \\
-\mathrm{N}''_{2n+1} &= \frac{1}{(2n+1)!}\{2\alpha S'_{2n+1} + \alpha T'_{2n} - (1+\alpha^2)\,C'_{2n+1} + (1-\alpha^2)\,I'_{2n+1}\} \\
&\qquad + \frac{1}{2(2n)!}\{C'_{2n} - I'_{2n} - \tfrac{1}{2}(C'_{2n-1} - D'_{2n-1} - I'_{2n-1} + J'_{2n-1}) - \alpha S'_{2n}\} \\
&\qquad - \frac{(1-2\sigma)}{(2n+1)!}\{\alpha S'_{2n} - C'_{2n} + I'_{2n} + n(C'_{2n-1} - I'_{2n-1}) - \tfrac{1}{2}(D'_{2n-1} - J'_{2n-1})\}, \\
-\mathrm{P}''_{2n} &= \frac{1}{2(2n+1)!}\{S'_{2n+1} - \tfrac{1}{2}(S'_{2n} - T'_{2n}) - \alpha(C'_{2n+1} + I'_{2n+1}) - (1-2\sigma)\,S'_{2n}\}, \\
-\mathrm{P}''_{2n+1} &= \frac{1}{2(2n+2)!}\{C'_{2n+2} - I'_{2n+2} - \tfrac{1}{2}(C'_{2n+1} - D'_{2n+1} - I'_{2n+1} + J'_{2n+1}) - \alpha S'_{2n+2} \\
&\qquad - (1-2\sigma)\,(C'_{2n+1} - I'_{2n+1})\}.
\end{aligned}\right\} \tag{5·26}$$

The second expansion for a longitudinal force has therefore the form

$$\begin{aligned}
\chi' = \frac{\mathrm{P}b}{4\pi(1-\sigma)}\Big\{&(1-2\sigma)\sum_{n=0}^{\infty}\rho^n[(l_n + m_n\rho^2)\cos n\theta + (n_n + p_n\rho^2)\sin n\theta] \\
&+ 2(1-\sigma)\sum_{n=0}^{\infty}\rho^n[(l'_n + m'_n\rho^2)\cos n\theta + (n'_n + p'_n\rho^2)\sin n\theta] \\
&+ \sum_{n=0}^{\infty}\rho^n[\{(\mathrm{L}'_n + \mathrm{L}''_n) + (\mathrm{M}'_n + \mathrm{M}''_n)\,\rho^2\}\cos n\theta \\
&\qquad + \{(\mathrm{N}'_n + \mathrm{N}''_n) + (\mathrm{P}'_n + \mathrm{P}''_n)\,\rho^2\}\sin n\theta]\Big\}.
\end{aligned} \tag{5·27}$$

In the first few coefficients of the series we find divergent integrals of the same character as those found in (5·14). These have been investigated and no unusual combination is found. We omit the details here as also the discussion of those of the two following expansions.

### (c) *Transverse force at (o, h) expanded about (o, h)*

The transverse force is one acting perpendicular to the parallel edges of the strip, i.e. in direction $\overrightarrow{\mathrm{OY}}$. If it acts at the point $(o, h)$, it will have as its stress function

$$\chi'' = \frac{\mathrm{P}b}{4\pi(1-\sigma)}\{(1-2\sigma)\,(\eta-\alpha)\log\rho + 2(1-\sigma)\,\xi\theta + \Phi''\}, \tag{5·28}$$

where $$\Phi'' = \chi''_1 + \chi''_2 + \chi''_3 + \chi''_4,$$

and

$$\left.\begin{aligned} \chi_1'' &= \int_0^\infty \frac{u\eta s\mathrm{S}-(s+uc)\,\mathrm{C}}{u^2\Sigma}(\mathrm{P}_1-\mathrm{P}_2)\cos u\xi\,du, \\ \chi_2'' &= \int_0^\infty \frac{\eta c\mathrm{S}-s\mathrm{C}}{u\Sigma}(\mathrm{P}_1'-\mathrm{P}_2')\cos u\xi\,du, \\ \chi_3'' &= \int_0^\infty \frac{u\eta c\mathrm{C}-(c+us)\,\mathrm{S}}{u^2\Sigma'}(\mathrm{P}_1+\mathrm{P}_2)\cos u\xi\,du, \\ \chi_4'' &= \int_0^\infty \frac{\eta s\mathrm{C}-c\mathrm{S}}{u\Sigma'}(\mathrm{P}_1'+\mathrm{P}_2')\cos u\xi\,du; \end{aligned}\right\} \tag{5·29}$$

with

$$\left.\begin{aligned} \mathrm{P}_1 &= [2(1-\sigma)+u(1-\alpha)]\,e^{-u(1-\alpha)}, \\ \mathrm{P}_2 &= [2(1-\sigma)+u(1+\alpha)]\,e^{-u(1+\alpha)}, \\ \mathrm{P}_1' &= [(1-2\sigma)+u(1-\alpha)]\,e^{-u(1-\alpha)}, \\ \mathrm{P}_2' &= [(1-2\sigma)+u(1+\alpha)]\,e^{-u(1+\alpha)}. \end{aligned}\right\} \tag{5·30}$$

The reduction of $\Phi''$ to its final series form is carried out in the same way as in case ($a$). We omit the details of this and give the stress function in its required expansion. This is

$$\chi'' = \frac{\mathrm{P}b}{4\pi(1-\sigma)}\Big\{(1-2\sigma)\,\rho\cos\theta\log\rho+2(1-\sigma)\,\theta\rho\sin\theta + \sum_{n=1}^{\infty}[(\mathrm{C}_n'+\mathrm{C}_n'')+(\mathrm{D}_n'+\mathrm{D}_n'')\,\rho^2]\,\rho^n\cos n\theta\Big\}. \tag{5·31}$$

The coefficients in the series are

$$\left.\begin{aligned} \mathrm{C}_{2n}' &= \frac{1}{(2n)!}\{2\alpha\mathrm{C}_{2n}-(n-1)\,\alpha\mathrm{C}_{2n-1}-\alpha\mathrm{D}_{2n-1}-(1+\alpha^2)\,\mathrm{S}_{2n}+(n-1)\,\mathrm{S}_{2n-1} \\ &\qquad +\tfrac{1}{2}(n-1)\,(\mathrm{S}_{2n-2}-\mathrm{T}_{2n-2})-n\alpha\mathrm{I}_{2n-1} \\ &\qquad -(1-2\sigma)\,[\alpha(\mathrm{I}_{2n-1}-\mathrm{C}_{2n-1})+\mathrm{S}_{2n-1}-(n-\tfrac{1}{2})\,\mathrm{S}_{2n-2}-\tfrac{1}{2}\mathrm{T}_{2n-2}]\}, \\ \mathrm{C}_{2n+1}' &= \frac{1}{(2n+1)!}\{(1-\alpha^2)\,\mathrm{I}_{2n+1}-(1+\alpha^2)\,\mathrm{C}_{2n+1}+(n-\tfrac{1}{2})\,\tfrac{1}{2}(\mathrm{C}_{2n-1}-\mathrm{D}_{2n-1}-\mathrm{I}_{2n-1}+\mathrm{J}_{2n-1}) \\ &\qquad +(n-\tfrac{1}{2})\,(\mathrm{C}_{2n}-\mathrm{I}_{2n})+\alpha\mathrm{S}_{2n+1}-(n-\tfrac{1}{2})\,\alpha\mathrm{S}_{2n}-\alpha\mathrm{T}_{2n} \\ &\qquad +(1-2\sigma)\,[\alpha\mathrm{S}_{2n}-(\mathrm{C}_{2n}-\mathrm{I}_{2n})+n(\mathrm{C}_{2n-1}-\mathrm{I}_{2n-1})+\tfrac{1}{2}(\mathrm{D}_{2n-1}-\mathrm{J}_{2n-1})]\}, \\ \mathrm{D}_{2n}' &= \frac{1}{2(2n+1)!}\{-\alpha(\mathrm{C}_{2n+1}+\mathrm{I}_{2n+1})+\mathrm{S}_{2n+1}+\tfrac{1}{2}(\mathrm{S}_{2n}-\mathrm{T}_{2n})+(1-2\sigma)\,\mathrm{S}_{2n}\}, \\ \mathrm{D}_{2n+1}' &= \frac{1}{2(2n+2)!}\{\mathrm{C}_{2n+2}-\mathrm{I}_{2n+2}+\tfrac{1}{2}(\mathrm{C}_{2n+1}-\mathrm{D}_{2n+1}-\mathrm{I}_{2n+1}+\mathrm{J}_{2n+1}) \\ &\qquad +(1-2\sigma)\,(\mathrm{C}_{2n+1}-\mathrm{I}_{2n+1})\}; \end{aligned}\right\} \tag{5·32}$$

$$\left.\begin{aligned}
C''_{2n} &= \frac{1}{(2n)!}\{(n-1)\,\alpha C'_{2n-1}+n\alpha I'_{2n-1}+2\alpha C'_{2n-2}+\alpha D'_{2n-1}-(1+\alpha^2)\,S'_{2n} \\
&\quad +(n-1)\,S'_{2n-1}+\tfrac{1}{2}(n-1)\,(S'_{2n-2}+T'_{2n-2}) \\
&\quad +(1-2\sigma)\,[(C'_{2n-1}+I'_{2n-1})-S'_{2n-1}+(n-\tfrac{1}{2})\,S'_{2n-2}-\tfrac{1}{2}T'_{2n-2}]\}, \\
C''_{2n+1} &= \frac{1}{(2n+1)!}\{-(1-\alpha^2)\,I'_{2n+1}-(1+\alpha^2)\,C'_{2n+1}+(n-\tfrac{1}{2})\,(C'_{2n}+I'_{2n}) \\
&\quad +(n-\tfrac{1}{2})\,\tfrac{1}{2}(C'_{2n-1}+D'_{2n-1}+I'_{2n-1}+J'_{2n-1})+2\alpha S'_{2n+1}-(n-\tfrac{1}{2})\,\alpha S'_{2n}+\alpha T'_{2n} \\
&\quad +(1-2\sigma)\,[n(C'_{2n-1}+I'_{2n-1})+\alpha S'_{2n}-(C'_{2n}+I'_{2n})-\tfrac{1}{2}(D'_{2n-1}+J'_{2n-1})]\}, \\
D''_{2n} &= \frac{1}{2(2n+1)!}\{\alpha(I'_{2n+1}-C'_{2n+1})+S'_{2n+1}+\tfrac{1}{2}(S'_{2n}+T'_{2n})+(1-2\sigma)\,S'_{2n}\}, \\
D''_{2n+1} &= \frac{1}{2(2n+2)!}\{C'_{2n+2}+I'_{2n+2}+\tfrac{1}{2}(C'_{2n+1}+D'_{2n+1}+I'_{2n+1}+J'_{2n+1})-\alpha S'_{2n+2} \\
&\quad +(1-2\sigma)\,[C'_{2n+1}+I'_{2n+1})\}.
\end{aligned}\right\} \quad (5\cdot33)$$

(*d*) *Transverse force at* $(-a, -h)$ *expanded about* $(o, h)$

A force P acting at $(-a, -h)$ in direction $\overrightarrow{O'Y'}$ has the stress function

$$\chi''' = \frac{Pb}{4\pi(1-\sigma)}\{(1-2\sigma)\,(\eta'-\alpha)\log\rho'+2(1-\sigma)\,\xi'\theta'+\Phi'''\}, \quad (5\cdot34)$$

where the accented co-ordinates are those used in case (*b*). The function $\Phi'''$ is similar to $\Phi''$ but with $\xi'$, i.e. $-(\xi+\beta)$, for $\xi$, etc. In fact, the process of obtaining the required series follows closely that used in (*b*). We shall content ourselves with giving the final form. We find that

$$\begin{aligned}
\chi''' = \frac{Pb}{4\pi(1-\sigma)}\Big\{&(1-2\sigma)\sum_{n=0}^{\infty}\rho^n[(x_n+y_n\rho^2)\cos n\theta+(z_n+w_n\rho^2)\sin n\theta] \\
&+2(1-\sigma)\sum_{n=0}^{\infty}\rho^n[(x'_n+y'_n\rho^2)\cos n\theta+(z'_n+w'_n\rho^2)\sin n\theta] \\
&+\sum_{n=0}^{\infty}\rho^n[\{(X'_n+X''_n)+(Y'_n+Y''_n)\,\rho^2\}\cos n\theta \\
&+\{(Z'_n+Z''_n)+(W'_n+W''_n)\,\rho^2\}\sin n\theta]\Big\},
\end{aligned} \quad (5\cdot35)$$

where

$$\left.\begin{aligned}
x_n &= (-)^n\{2(n-2)\,\alpha\cos n\phi+n\beta\sin n\phi\}/\kappa^n\,2n(n-1), \\
y_n &= (-)^{n+1}\{\cos(n+1)\,\phi\}/\kappa^{n+1}\,2(n+1), \\
z_n &= (-)^n\{2(n-2)\,\alpha\sin n\phi-n\beta\cos n\phi\}/\kappa^n\,2n(n-1), \\
w_n &= (-)^{n+1}\{\sin(n+1)\,\phi\}/\kappa^{n+1}\,2(n+1);
\end{aligned}\right\} \quad (5\cdot36)$$



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$$\left.\begin{aligned} x'_n &= (-)^{n+1}\{(n-2)\,\beta\cos n\phi+2n\alpha\sin n\phi\}/\kappa^n\,2n(n-1),\\ y'_n &= -y_n,\\ z'_n &= (-)^{n+1}\{(n-2)\,\beta\sin n\phi-2n\alpha\cos n\phi\}/\kappa^n\,2n(n-1),\\ w'_n &= -w_n; \end{aligned}\right\} \tag{5·37}$$

$$\left.\begin{aligned} -\mathrm{X}'_{2n} &= \frac{1}{(2n)!}\{2\alpha C_{2n}-(n-1)\,\alpha C_{2n-1}-\alpha D_{2n-1}-(1+\alpha^2)\,S_{2n}+(n-1)\,S_{2n-1}\\ &\quad+\tfrac{1}{2}(n-1)\,(S_{2n-2}-T_{2n-2})-n\alpha I_{2n-1}\\ &\quad-(1-2\sigma)\,[\alpha(I_{2n-1}-C_{2n-1})+S_{2n-1}-(n-\tfrac{1}{2})\,S_{2n-2}-\tfrac{1}{2}T_{2n-2}]\},\\ -\mathrm{X}'_{2n+1} &= \frac{1}{(2n+1)!}\{(1-\alpha^2)\,I_{2n+1}-(1+\alpha^2)\,C_{2n+1}+(n-\tfrac{1}{2})\,(C_{2n}-I_{2n})\\ &\quad+\tfrac{1}{2}(n-\tfrac{1}{2})\,(C_{2n-1}-D_{2n-1}-I_{2n-1}+J_{2n-1})+\alpha S_{2n+1}-(n-\tfrac{1}{2})\,\alpha S_{2n}-\alpha T_{2n}\\ &\quad+(1-2\sigma)\,[\alpha S_{2n}-(C_{2n}-I_{2n})+n(C_{2n-1}-I_{2n-1})+\tfrac{1}{2}(D_{2n-1}-J_{2n-1})]\},\\ -\mathrm{Y}'_{2n} &= \frac{1}{2(2n+1)!}\{-\alpha(C_{2n+1}+I_{2n+1})+S_{2n+1}+\tfrac{1}{2}(S_{2n}-T_{2n})+(1-2\sigma)\,S_{2n}\},\\ -\mathrm{Y}'_{2n+1} &= \frac{1}{2(2n+2)!}\{C_{2n+2}-I_{2n+2}+\tfrac{1}{2}(C_{2n+1}-D_{2n+1}-I_{2n+1}+J_{2n+1})\\ &\quad+(1-2\sigma)\,(C_{2n+1}-I_{2n+1})\}; \end{aligned}\right\} \tag{5·38}$$

$$\left.\begin{aligned} -\mathrm{X}''_{2n} &= \frac{1}{(2n)!}\{(n-1)\,\alpha C'_{2n-1}+n\alpha I'_{2n-1}+2\alpha C'_{2n-2}+\alpha D'_{2n-1}-(1+\alpha^2)\,S'_{2n}\\ &\quad+(n-1)\,S'_{2n-1}+\tfrac{1}{2}(n-1)\,(S'_{2n-2}+T'_{2n-2})\\ &\quad+(1-2\sigma)\,[\alpha(C'_{2n-1}+I'_{2n-1})-S'_{2n-1}+(n-\tfrac{1}{2})\,S'_{2n-2}-\tfrac{1}{2}T'_{2n-2}]\},\\ -\mathrm{X}''_{2n+1} &= \frac{1}{(2n+1)!}\{-(1-\alpha^2)\,I'_{2n+1}-(1+\alpha^2)\,C'_{2n+1}+(n-\tfrac{1}{2})\,(C'_{2n}+I'_{2n})\\ &\quad+\tfrac{1}{2}(n-\tfrac{1}{2})\,(C'_{2n-1}+D'_{2n-1}+I'_{2n-1}+J'_{2n-1})+2\alpha S'_{2n+1}-(n-\tfrac{1}{2})\,\alpha S'_{2n}+\alpha T'_{2n}\\ &\quad+(1-2\sigma)\,[\alpha S'_{2n}-(C'_{2n}+I'_{2n})+n(C'_{2n-1}+I'_{2n-1})-\tfrac{1}{2}(D'_{2n-1}+J'_{2n-1})]\},\\ -\mathrm{Y}''_{2n} &= \frac{1}{2(2n+1)!}\{\alpha(I'_{2n+1}-C'_{2n+1})+S'_{2n+1}+\tfrac{1}{2}(S'_{2n}+T'_{2n})+(1-2\sigma)\,S'_{2n}\},\\ -\mathrm{Y}''_{2n+1} &= \frac{1}{2(2n+2)!}\{C'_{2n+2}+I'_{2n+2}+\tfrac{1}{2}(C'_{2n+1}+D'_{2n+1}+I'_{2n+1}+J'_{2n+1})-\alpha S'_{2n+2}\\ &\quad+(1-2\sigma)\,(C'_{2n+1}+I'_{2n+1})\}; \end{aligned}\right\} \tag{5·39}$$





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$$\left.\begin{aligned}
Z'_{2n} &= \frac{1}{(2n)!}\{(1-\alpha^2)\,\bar{I}_{2n}-(1+\alpha^2)\,\bar{C}_{2n}+(n-1)\,(\bar{C}_{2n-1}-\bar{I}_{2n-1})\\
&\quad+\tfrac{1}{2}(n-1)\,(\bar{C}_{2n-2}-\bar{D}_{2n-2}-\bar{I}_{2n-2}+\bar{J}_{2n-2})+2\alpha\bar{S}_{2n}-(n-1)\,\alpha\bar{S}_{2n-1}-\alpha\bar{T}_{2n-1}\\
&\quad-(1-2\sigma)\,[(n-\tfrac{1}{2})\,(\bar{C}_{2n-2}-\bar{I}_{2n-2})-(\bar{C}_{2n-1}-\bar{I}_{2n-1})+\tfrac{1}{2}(\bar{D}_{2n-2}-\bar{J}_{2n-2})+\alpha\bar{S}_{2n-1}]\};\\
Z'_{2n+1} &= \frac{1}{(2n+1)!}\{2\alpha\bar{C}_{2n+1}-(n-\tfrac{1}{2})\,\bar{C}_{2n}-\alpha\bar{D}_{2n}-(n-\tfrac{1}{2})\,\alpha\bar{I}_{2n}-(1+\alpha^2)\,\bar{S}_{2n+1}\\
&\quad+(n-\tfrac{1}{2})\,\bar{S}_{2n}+\tfrac{1}{2}(n-\tfrac{1}{2})\,(\bar{S}_{2n-1}-\bar{T}_{2n-1})\\
&\quad-(1-2\sigma)\,[\alpha(\bar{I}_{2n}-\bar{C}_{2n})-n\bar{S}_{2n-1}+\bar{S}_{2n}-\tfrac{1}{2}\bar{T}_{2n-1}]\},\\
W'_{2n} &= \frac{1}{2(2n+1)!}\{\bar{C}_{2n+1}-\bar{I}_{2n+1}+\tfrac{1}{2}(\bar{C}_{2n}-\bar{D}_{2n}-\bar{I}_{2n}+\bar{J}_{2n})-\alpha\bar{S}_{2n+1}+(1-2\sigma)\,(\bar{C}_{2n}-\bar{I}_{2n})\},\\
W'_{2n+1} &= \frac{1}{2(2n+2)!}\{-\alpha(\bar{C}_{2n+2}+\bar{I}_{2n+2})+\bar{S}_{2n+2}+\tfrac{1}{2}(\bar{S}_{2n+1}-\bar{T}_{2n+1})+(1-2\sigma)\,\bar{S}_{2n+1}\};
\end{aligned}\right\}\quad(5\cdot40)$$

$$\left.\begin{aligned}
Z''_{2n} &= \frac{1}{(2n)!}\{-(1+\alpha^2)\,\bar{C}'_{2n}+(n-1)\,(\bar{C}'_{2n-1}+\bar{I}'_{2n-1})+\tfrac{1}{2}(n-1)\,(\bar{C}'_{2n-2}+\bar{D}'_{2n-2}\\
&\quad+\bar{I}'_{2n-2}+\bar{J}'_{2n-2})-(1-\alpha^2)\,\bar{I}'_{2n}+2\alpha\bar{S}'_{2n}+(n-1)\,\alpha\bar{S}'_{2n-1}+\alpha\bar{T}'_{2n-1}\\
&\quad+(1-2\sigma)\,[\bar{C}'_{2n-1}+\bar{I}'_{2n-1}+(n+\tfrac{1}{2})\,(\bar{C}'_{2n-2}+\bar{I}'_{2n-2})+\tfrac{1}{2}(\bar{D}'_{2n-2}+\bar{J}'_{2n-2})+\alpha\bar{S}'_{2n-1}]\},\\
Z''_{2n+1} &= \frac{1}{(2n+1)!}\{2\alpha\bar{C}'_{2n+1}-(n-\tfrac{1}{2})\,\bar{C}'_{2n}+\alpha\bar{D}'_{2n}+(n+\tfrac{1}{2})\,\alpha\bar{I}'_{2n}-(1+\alpha^2)\,\bar{S}'_{2n+1}\\
&\quad+(n-\tfrac{1}{2})\,\bar{S}'_{2n}+\tfrac{1}{2}(n-\tfrac{1}{2})\,(\bar{S}'_{2n-1}+\bar{T}'_{2n-1})\\
&\quad+(1-2\sigma)\,[\alpha(\bar{C}'_{2n}+\bar{I}'_{2n})+\bar{S}'_{2n}+n\bar{S}'_{2n-1}-\tfrac{1}{2}\bar{T}'_{2n-1}]\},\\
W''_{2n} &= \frac{1}{2(2n+1)!}\{\bar{C}'_{2n+1}+\bar{I}'_{2n+1}+\tfrac{1}{2}(\bar{C}'_{2n}+\bar{D}'_{2n}+\bar{I}'_{2n}+\bar{J}'_{2n})-\alpha\bar{S}'_{2n+1}+(1-2\sigma)\,(\bar{C}'_{2n}+\bar{I}'_{2n})\},\\
W''_{2n+1} &= \frac{1}{2(2n+2)!}\{\alpha(\bar{I}'_{2n+2}-\bar{C}'_{2n+2})+\bar{S}'_{2n+2}+\tfrac{1}{2}(\bar{S}'_{2n+1}+\bar{T}'_{2n+1})+(1-2\sigma)\,\bar{S}'_{2n+1}\}.
\end{aligned}\right\}\quad(5\cdot41)$$

## 6. A pair of circles in a strip

The general case when the circles are arbitrarily placed in a strip can be considered with the aid of the expansions obtained in the previous sections, provided there is symmetry about a point on the axis of the strip. The algebra, although straightforward, is very heavy, and we shall content ourselves with the investigation of two special problems. The first of these is when the centres of the circles lie on the axis.

### (a) *First symmetrical case*

Let the centres be at the points $(0, 0)$, $(-a, 0)$, the strip as before being defined by $y = \pm b$. Using the notation of the previous sections, the centres are given by $\xi = 0$, $\xi = -\beta$.

We start with the stress function for two forces P acting at the centres as shown. This function is

$$\chi_0 = \chi - \chi', \qquad (6\cdot1)$$

where $\chi$ is given by equation (5·14) and $\chi'$ by (5·27) after putting $\alpha$ zero.

Consider $\chi$ first. Since $\alpha$ is zero, the coefficients (5·12) and (5·13) have simplified values while the integrals (5·10) are related by the equations

$$\mathrm{C}_n = \mathrm{I}_n, \quad \mathrm{D}_n = \mathrm{J}_n, \quad \mathrm{S}_n = \mathrm{T}_n = 0. \tag{6 2}$$

Consequently all the coefficients vanish except

$$\left.\begin{aligned} \mathrm{A}'_{2n+1} &= \frac{1}{(2n+1)!}\{-2\mathrm{I}_{2n+1}+(2n+1)\,(\mathrm{I}_{2n}-\tfrac{1}{2}\mathrm{I}_{2n-1}-\tfrac{1}{2}\mathrm{J}_{2n-1}) \\ &\qquad +(1-2\sigma)\,(2\mathrm{I}_{2n}-2n\mathrm{I}_{2n-1}-\mathrm{J}_{2n-1})\}, \\ \mathrm{B}'_{2n+1} &= \frac{1}{(2n+2)!}\{\mathrm{I}_{2n+2}-\tfrac{1}{2}(\mathrm{I}_{2n+1}+\mathrm{J}_{2n+1})-(1-2\sigma)\,\mathrm{I}_{2n+1}\}. \end{aligned}\right\} \tag{6·3}$$

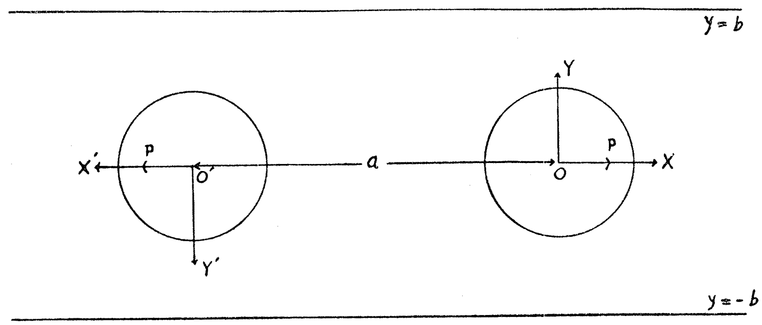


FIG. 8

Taking the simplified $\chi'$ in the same way, we find that $\kappa = \beta$ and $\phi = \frac{1}{2}\pi$ and the appropriate values of the coefficients are

$$\left.\begin{aligned} l_{2n} &= \frac{(-)^{n+1}\,(n-1)}{2n(2n-1)\,\beta^{2n-1}}, \qquad m_{2n} = m'_{2n} = \frac{(-)^n}{2(2n+1)\,\beta^{2n+1}}, \\ n_{2n+1} &= \frac{(-)^n\,(2n-1)}{4n(2n+1)\,\beta^{2n}}, \qquad p_{2n+1} = p'_{2n+1} = \frac{(-)^{n+1}}{4(n+1)\,\beta^{2n+2}}, \\ l'_{2n} &= \frac{(-)^{n+1}}{2(2n-1)\,\beta^{2n-1}}, \qquad n'_{2n+1} = \frac{(-)^n}{4n\beta^{2n}}; \end{aligned}\right\} \tag{6·4}$$

the others vanishing.

The integrals (5·22) have their particular values and the remaining coefficients derived from (5·23)–(5·26) become zero with the exception of

$$\left.\begin{aligned} \mathrm{L}'_{2n} &= \frac{1}{(2n)!}\{-2\bar{I}_{2n}+2n\bar{I}_{2n-1}-n(\bar{I}_{2n-2}+\bar{J}_{2n-2}) \\ &\qquad +(1-2\sigma)\,(2\bar{I}_{2n-1}-\bar{J}_{2n-2}-[2n-1]\,\bar{I}_{2n-2})\}, \\ \mathrm{M}'_{2n} &= \frac{1}{(2n+1)!}\{\bar{I}_{2n+1}-\tfrac{1}{2}(\bar{I}_{2n}+\bar{J}_{2n})-(1-2\sigma)\,\bar{I}_{2n}\}, \\ \mathrm{N}'_{2n+1} &= \frac{1}{(2n+1)!}\{-2I_{2n+1}+(2n+1)\,[I_{2n}-\tfrac{1}{2}(I_{2n-1}-J_{2n-1})] \\ &\qquad +(1-2\sigma)\,(2I_{2n}-2nI_{2n-1}-J_{2n-1})\}, \\ \mathrm{P}'_{2n+1} &= \frac{1}{(2n+2)!}\{I_{2n+2}-\tfrac{1}{2}(I_{2n+1}+J_{2n+1})-(1-2\sigma)\,I_{2n+1}\}; \end{aligned}\right\} \tag{6·5}$$

With these values the stress function becomes

$$\chi_0 = \frac{Pb}{4\pi(1-\sigma)}\Big\{(1-2\sigma)\,\rho\sin\theta\log\rho+2(1-\sigma)\,\theta\rho\cos\theta$$
$$+\sum_{n=0}^{\infty}[(a_{2n}+a'_{2n}\rho^2)\,\rho^{2n}\cos 2n\theta+(b_{2n+1}+b'_{2n+1}\rho^2)\,\rho^{2n+1}\sin(2n+1)\,\theta]\Big\}, \qquad (6\cdot6)$$

where
$$\left.\begin{aligned} a_{2n} &= (1-2\sigma)\,l_{2n}+2(1-\sigma)\,l'_{2n}+L'_{2n},\\ a'_{2n} &= (1-2\sigma)\,m_{2n}+2(1-\sigma)\,m'_{2n}+M'_{2n},\\ b_{2n+1} &= (1-2\sigma)\,n_{2n+1}+2(1-\sigma)\,n'_{2n+1}+A'_{2n+1}+N'_{2n+1},\\ b'_{2n+1} &= (1-2\sigma)\,p_{2n+1}+2(1-\sigma)\,p'_{2n+1}+B'_{2n+1}+P'_{2n+1}.\end{aligned}\right\} \qquad (6\cdot7)$$

$\chi_0$ is a biharmonic function, containing the elastic constant $\sigma$. It is therefore biharmonic, of the correct invariance and satisfying the straight edge boundary condition for any particular value of $\sigma$. We must, however, remember the significance of $\sigma$ when dealing with stress functions which might produce multiple value expressions for the displacements. If $\chi_0$ is used, $\sigma$ must be retained; but we shall derive from this function a set of biharmonic functions which, whatever the value of $\sigma$, give single valued expressions for the displacements. Hence $\sigma$ may be given special values. It is simpler to specify the value of $\sigma$ and then derive the functions. Two sets of functions will be required.

To obtain the first set let $Pb/2\pi = 1$ and also let $\sigma\to\infty$. Then define

$$U_0 = -\frac{\partial\chi_0}{\partial\xi}, \quad U_s = \frac{\partial^s U_0}{\partial\xi^s}, \quad (s\geqslant 1). \qquad (6\cdot8)$$

After some reduction, and omitting a multiplying factor, we find that

$$U_0 = -\log\rho+\sum_{n=0}^{\infty}({}^0A_{2n}+{}^0A'_{2n}\rho^2)\,\rho^{2n}\cos 2n\theta$$
$$+\sum_{n=0}^{\infty}({}^0B_{2n+1}+{}^0B'_{2n+1}\rho^2)\,\rho^{2n+1}\sin(2n+1)\,\theta, \qquad (6\cdot9)$$

where
$$\left.\begin{aligned} {}^0A_{2n} &= -\frac{1}{(2n)!}\{2(I_{2n}+\mathit{I}_{2n})-(2n-1)\,(I_{2n-1}+\mathit{I}_{2n-1})-(J_{2n-1}+\mathit{J}_{2n-1})\}\\ &\qquad +\frac{(-)^{n+1}\,(2n-1)}{2n\beta^{2n}},\\ {}^0A'_{2n} &= \frac{1}{(2n+1)!}\{I_{2n+1}+\mathit{I}_{2n+1}\}+\frac{(-)^n}{\beta^{2n+2}},\\ {}^0B_{2n+1} &= \frac{1}{(2n+1)!}\{2\bar{I}_{2n+1}-2n\bar{I}_{2n}-\bar{J}_{2n}\}+\frac{(-)^n\,2n}{(2n+1)\,\beta^{2n+1}},\\ {}^0B'_{2n+1} &= -\frac{\bar{I}_{2n+2}}{(2n+2)!}+\frac{(-)^{n+1}}{\beta^{2n+3}}.\end{aligned}\right\} \qquad (6\cdot10)$$

When $s \geqslant 1$ we finally obtain, again omitting multiplying factors,

$$U_{2s} = \rho^{-2s}\cos 2s\theta + \sum_{n=0}^{\infty} ({}^{2s}A_{2n} + {}^{2s}A'_{2n}\rho^2)\,\rho^{2n}\cos 2n\theta + \sum_{n=0}^{\infty} ({}^{2s}B_{2n+1} + {}^{2s}B'_{2n+1}\rho^2)\,\rho^{2n+1}\sin(2n+1)\,\theta, \tag{6·11}$$

$$U_{2s+1} = \rho^{-2s-1}\sin(2s+1)\,\theta + \sum_{n=0}^{\infty} ({}^{2s+1}A_{2n} + {}^{2s+1}A'_{2n}\rho^2)\,\rho^{2n}\cos 2n\theta + \sum_{n=0}^{\infty} ({}^{2s+1}B_{2n+1} + {}^{2s+1}B'_{2n+1}\rho^2)\,\rho^{2n+1}\sin(2n+1)\,\theta. \tag{6·12}$$

The coefficients in the even functions are

$$\left.\begin{aligned}
{}^{2s}A_{2n} &= -\frac{1}{(2s-1)!\,(2n)!}\{2(I_{2n+2s}+\mathit{I}_{2n+2s}) - (2n-1)\,(I_{2n+2s-1}+\mathit{I}_{2n+2s-1}) \\
&\qquad - (J_{2n+2s-1}+\mathit{J}_{2n+2s-1})\} + (-)^{n+s-1}\binom{2n+2s-1}{2n}\frac{2n-1}{\beta^{2n+2s}}, \\
{}^{2s}A'_{2n} &= \frac{1}{(2s-1)!\,(2n+1)!}\{I_{2n+2s+1}+\mathit{I}_{2n+2s+1}\} + (-)^{n+s}\binom{2n+2s+1}{2n+2}\frac{2n+2}{\beta^{2n+2s+2}}, \\
{}^{2s}B_{2n+1} &= \frac{1}{(2s-1)!\,(2n+1)!}\{2\bar{I}_{2n+2s+1} - 2n\bar{I}_{2n+2s} - \bar{J}_{2n+2s}\} \\
&\qquad + (-)^{n+s}\binom{2n+2s}{2n+1}\frac{2n}{\beta^{2n+2s+1}}, \\
{}^{2s}B'_{2n+1} &= -\frac{\bar{I}_{2n+2s+2}}{(2s-1)!\,(2n+1)!} + (-)^{n+s+1}\binom{2n+2s+2}{2n+3}\frac{2n+3}{\beta^{2n+2s+3}}.
\end{aligned}\right\} \tag{6·13}$$

In the odd functions

$$\left.\begin{aligned}
{}^{2s+1}A_{2n} &= -\frac{1}{(2s)!\,(2n)!}\{2\bar{I}_{2n+2s+1} - (2n-1)\,\bar{I}_{2n+2s} - \bar{J}_{2n+2s}\} \\
&\qquad + (-)^{n+s-1}\binom{2n+2s}{2n}\frac{2n-1}{\beta^{2n+2s+1}}, \\
{}^{2s+1}A'_{2n} &= \frac{\bar{I}_{2n+2s+2}}{(2s)!\,(2n+1)!} + (-)^{n+s}\binom{2n+2s+2}{2n+2}\frac{2n+2}{\beta^{2n+2s+3}}, \\
{}^{2s+1}B_{2n+1} &= -\frac{1}{(2s)!\,(2n+1)!}\{2(I_{2n+2s+2}+\mathit{I}_{2n+2s+2}) - 2n(I_{2n+2s+1}+\mathit{I}_{2n+2s+1}) \\
&\qquad - (J_{2n+2s+1}+\mathit{J}_{2n+2s+1})\} + (-)^{n+s}\binom{2n+2s+1}{2n+1}\frac{2n}{\beta^{2n+2s+2}}, \\
{}^{2s+1}B'_{2n+1} &= \frac{1}{(2s)!\,(2n+2)!}\{I_{2n+2s+3}+\mathit{I}_{2n+2s+3}\} + (-)^{n+s+1}\binom{2n+2s+3}{2n+3}\frac{2n+3}{\beta^{2n+2s+4}}.
\end{aligned}\right\} \tag{6·14}$$

The second set of functions may be obtained by taking $\sigma = \frac{1}{2}$.

If then
$$V_0 = 2\frac{\partial \chi_0}{\partial \xi}, \quad V_s = \frac{\partial^s V_0}{\partial \xi^s}, \quad (s \geq 1), \tag{6·15}$$

we have
$$V_0 = \cos 2\theta + \sum_{n=0}^{\infty} ({}^0C_{2n} + {}^0C'_{2n}\rho^2)\,\rho^{2n}\cos 2n\theta + \sum_{n=0}^{\infty} ({}^0D_{2n+1} + {}^0D'_{2n+1}\rho^2)\,\rho^{2n}\sin(2n+1)\theta, \tag{6·16}$$

where
$$\left.\begin{aligned}
{}^0C_{2n} &= \frac{1}{(2n)!}\{-4(I_{2n+1}+\mathcal{I}_{2n+1}) \\
&\quad + 2n[2(I_{2n}+\mathcal{I}_{2n}) - (I_{2n-1}+\mathcal{I}_{2n-1}) - (J_{2n-1}+\mathcal{J}_{2n-1})]\} + \frac{(-)^n}{\beta^{2n}}, \\
{}^0C'_{2n} &= \frac{1}{(2n+1)!}\{2(I_{2n+2}+\mathcal{I}_{2n+2}) - (I_{2n+1}+\mathcal{I}_{2n+1}) - (J_{2n+1}+\mathcal{J}_{2n+1})\} + \frac{(-)^{n+1}}{\beta^{2n+2}}, \\
{}^0D_{2n+1} &= -\frac{1}{(2n+1)!}\{-4\bar{\mathcal{I}}_{2n+2} + 2n(2\bar{\mathcal{I}}_{2n+1} - \bar{\mathcal{I}}_{2n} - \bar{\mathcal{J}}_{2n})\} + \frac{(-)^{n+1}}{\beta^{2n+1}}, \\
{}^0D'_{2n+1} &= -\frac{1}{(2n+2)!}\{2\bar{\mathcal{I}}_{2n+3} - (\bar{\mathcal{I}}_{2n+2} + \bar{\mathcal{J}}_{2n+2})\} + \frac{(-)^n}{\beta^{2n+3}}.
\end{aligned}\right\} \tag{6·17}$$

Differentiation with respect to $\xi$ leads to the general functions which are

$$V_{2s} = \rho^{-2s}[\cos(2s+2)\,\theta + \cos 2s\theta] + \sum_{n=0}^{\infty} ({}^{2s}C_{2n} + {}^{2s}C'_{2n}\rho^2)\,\rho^{2n}\cos 2n\theta + \sum_{n=0}^{\infty} ({}^{2s}D_{2n+1} + {}^{2s}D'_{2n+1}\rho^2)\,\rho^{2n+1}\sin(2n+1)\,\theta, \tag{6·18}$$

$$V_{2s+1} = \rho^{-2s-1}[\sin(2s+3)\,\theta + \sin(2s+1)\,\theta] + \sum_{n=0}^{\infty} ({}^{2s+1}C_{2n} + {}^{2s+1}C'_{2n}\rho^2)\,\rho^{2n}\cos 2n\theta + \sum_{n=0}^{\infty} ({}^{2s+1}D_{2n+1} + {}^{2s+1}D'_{2n+1}\rho^2)\,\rho^{2n+1}\sin(2n+1)\,\theta. \tag{6·19}$$

The new coefficients are, in the even functions

$$\left.\begin{aligned}
{}^{2s}C_{2n} &= \frac{1}{(2s)!\,(2n)!}\{-4(I_{2n+2s+1}+\mathcal{I}_{2n+2s+1}) + 2n[2(I_{2n+2s}+\mathcal{I}_{2n+2s}) \\
&\quad - (I_{2n+2s-1}+\mathcal{I}_{2n+2s-1}) - (J_{2n+2s-1}+\mathcal{J}_{2n+2s-1})]\} \\
&\quad + (-)^{n+s}\binom{2n+2s-1}{2n-1}\frac{1}{\beta^{2n+2s}}, \\
{}^{2s}C'_{2n} &= \frac{1}{(2s)!\,(2n+1)!}\{2(I_{2n+2s+2}+\mathcal{I}_{2n+2s+2}) - (I_{2n+2s+1}+\mathcal{I}_{2n+2s+1}) \\
&\quad - (\mathcal{J}_{2n+2s+1}+\mathcal{J}_{2n+2s+1})\} + (-)^{n+s+1}\binom{2n+2s+1}{2n+1}\frac{1}{\beta^{2n+2s+2}}, \\
{}^{2s}D_{2n+1} &= -\frac{1}{(2s)!\,(2n+1)!}\{-4\bar{\mathcal{I}}_{2n+2s+2} + 2n(2\bar{\mathcal{I}}_{2n+2s+1} - \bar{\mathcal{I}}_{2n+2s} - \bar{\mathcal{J}}_{2n+2s})\} \\
&\quad + (-)^{n+s+1}\binom{2n+2s}{2n}\frac{1}{\beta^{2n+2s+1}}, \\
{}^{2s}D'_{2n+1} &= -\frac{1}{(2s)!\,(2n+2)!}\{2\bar{\mathcal{I}}_{2n+2s+3} - (\bar{\mathcal{I}}_{2n+2s+2} + \bar{\mathcal{J}}_{2n+2s+2})\} \\
&\quad + (-)^{n+s}\binom{2n+2s+2}{2n+2}\frac{1}{\beta^{2n+2s+3}}.
\end{aligned}\right\} \tag{6·20}$$

While for the odd functions

$$\left.\begin{aligned}
{}^{2s+1}\mathrm{C}_{2n} &= \frac{1}{(2s+1)!\,(2n)!}\{-4\bar{I}_{2n+2s+2}+(2n-1)\,(2\bar{I}_{2n+2s+1}-\bar{I}_{2n+2s}-\bar{J}_{2n+2s})\} \\
&\qquad +(-)^{n+s}\binom{2n+2s}{2n}\frac{2n}{\beta^{2n+2s+1}}, \\
{}^{2s+1}\mathrm{C}'_{2n} &= \frac{1}{(2s+1)!\,(2n+1)!}\{2\bar{I}_{2n+2s+3}-(\bar{I}_{2n+2s+2}+\bar{J}_{2n+2s+2})\} \\
&\qquad +(-)^{n+s+1}\binom{2n+2s+2}{2n+2}\frac{(2n+2)}{\beta^{2n+2s+3}}, \\
{}^{2s+1}\mathrm{D}_{2n+1} &= \frac{1}{(2s+1)!\,(2n+1)!}\{-4(\mathrm{I}_{2n+2s+3}+I_{2n+2s+3})+(2n-1)\,[2(\mathrm{I}_{2n+2s+2}+I_{2n+2s+2}) \\
&\qquad -(\mathrm{I}_{2n+2s+1}+I_{2n+2s+1})-(\mathrm{J}_{2n+2s+1}+J_{2n+2s+1})]\} \\
&\qquad +(-)^{n+s+1}\binom{2n+2s+1}{2n+1}\frac{(2n+1)}{\beta^{2n+2s+2}}, \\
{}^{2s+1}\mathrm{D}'_{2n+1} &= \frac{1}{(2s+1)!\,(2n+2)!}\{2(\mathrm{I}_{2n+2s+4}+I_{2n+2s+4})-(\mathrm{I}_{2n+2s+3}+I_{2n+2s+3}) \\
&\qquad -(\mathrm{J}_{2n+2s+3}+J_{2n+2s+3})\}+(-)^{n+s}\binom{2n+2s+3}{2n+3}\frac{(2n+3)}{\beta^{2n+2s+4}}.
\end{aligned}\right\} \quad (6{\cdot}21)$$

It is clear that the functions we have obtained are even in $y$ but odd in $x$ (if $x$, $y$ are co-ordinates with origin at the centre of symmetry). They are therefore suitable, with the addition of the appropriate $\chi$ giving the infinity conditions, for a problem such as that of a strip under pure tension at infinity. It is obvious that other sets of functions, e.g. even in $x$, odd in $y$, can be obtained should they be required by choosing the appropriate expansions from 5 ($c$) and 5 ($d$).

### ($b$) *Second symmetrical case*

We may next conveniently consider the other symmetrical case in which the circles are situated as shown in fig. 9.

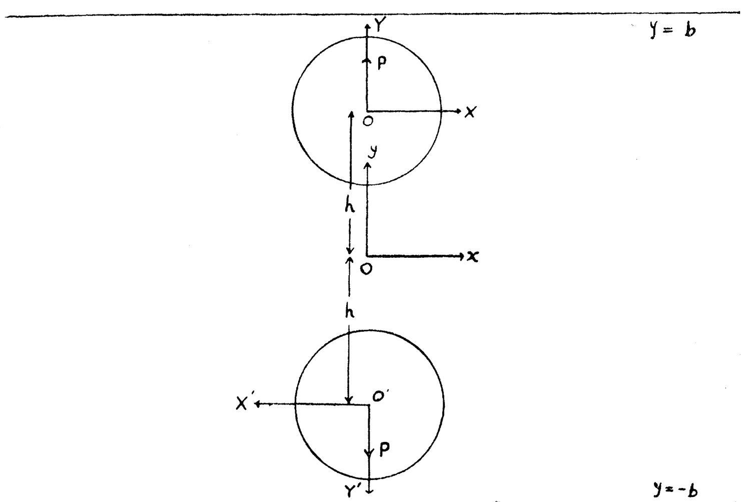


FIG. 9

Here we require, in the simplest case of symmetry of the boundary conditions, functions even in $x$. We therefore take as our fundamental stress function that giving forces P acting in directions OY, O′Y′, i.e.

$$\chi'_0 = \chi'' + \chi''',$$

where $\chi''$ is given by (5·31) and $\chi'''$ by (5·35) with $\beta = 0$. The evaluation of the coefficients of the required functions may be carried out as in the previous case and presents nothing new. Two sets of functions are required and may be derived with the same two special values of $\sigma$, namely, $\sigma = \frac{1}{2}$ and $\sigma \to \infty$.

## 7. Two pairs of circles in a strip

There remains one more configuration of circles in a strip which may be considered with the expansions we have developed. The circles, four in number, are situated in the strip as shown in fig. 10. The algebra is necessarily heavy, for we have no simplification such as the vanishing of $\alpha$ or $\beta$. The work would, however, be straightforward and presents no serious difficulty. We shall content ourselves by indicating how the fundamental functions can be set up. Four sets of functions will be required, each set being derived from one or other of two stress functions $\chi$ or $\chi_1$.

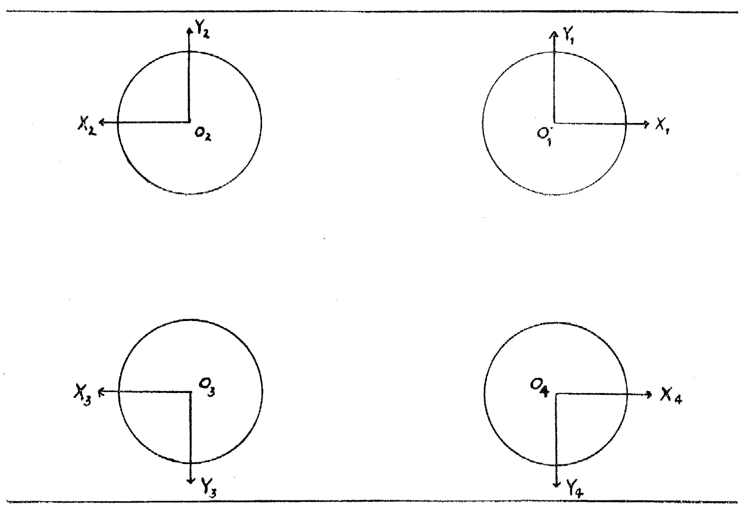


Fig. 10

First let
$$\chi = \chi^{(1)} - \chi^{(2)} - \chi^{(3)} + \chi^{(4)}, \tag{7·1}$$

where $\chi^{(r)}$ is the contribution from the singularity at $O_r$, $r = 1, 2, 3, 4$. Suppose forces to act at the centres in directions $\overrightarrow{O_rX_r}$.

$\chi^{(1)}$ is given by (5·14).

$\chi^{(2)}$ is obtained from (5·27) when $\alpha$ is replaced by $-\alpha$ in the various coefficients, and $\theta$ is replaced by $\pi - \theta$.

$\chi^{(3)}$ is given by (5·27).

$\chi^{(4)}$ is obtained from (5·27) by changing the sign of $\theta$ throughout, with $\beta = 0$.

The second function is obtained from the expansions of the transverse forces as follows:

Let
$$\chi_1 = \chi_1^{(1)} - \chi_1^{(2)} + \chi_1^{(3)} + \chi_1^{(4)}, \tag{7·2}$$
where

$\chi_1^{(1)}$ is given by (5·31).

$\chi_1^{(2)}$ is obtained from (5·35) with $-\alpha$ for $\alpha$ and $\pi-\theta$ for $\theta$.

$\chi_1^{(3)}$ is given by (5·35).

$\chi_1^{(4)}$ is obtained from (5·35) with $\beta = 0$ and $-\theta$ for $\theta$.

The derivation of the functions is the same as in the previous section and the same two special values of $\sigma$ will be found suitable.

## Conclusion

Although the functions set up have been obtained with the object of providing machinery for the solution of problems in generalized plane stress, they are of course suitable for the solution of the biharmonic equation in general. The only restriction is the symmetry of the boundaries and the conditions on them. When the boundaries are partly straight lines, the functions obtained will have to be adjusted to the appropriate conditions of the new problem, e.g. the slow motion of a viscous fluid (cf. Howland and Knight 1932).

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## Summary

A number of solutions of the biharmonic equation have been obtained, mostly in connexion with the problems of generalized plane stress, when the boundaries have consisted of circles and straight lines. No general method of solution can be given but the methods which the present authors have used in certain cases are here extended to a group of problems. The paper deals with circles in (*a*) the infinite plane, (*b*) a strip bounded by parallel lines. The circles, their number and relative positions, are restricted by an invariancy condition, which demands that the circles and their boundary conditions remain invariant under one or more of a group of transformations and/or reflexions.

In (*a*) the configurations have the boundaries (i) one pair of circles, (ii) two pairs of circles, (iii) a double infinite row of circles. While in (*b*) (i) one pair, and (ii) two pairs of circles are dealt with. These together with solutions previously published complete the group of problems to which the method is applicable.

No numerical work is included but the expansions of the necessary functions have been determined. They may be used for any problem where the biharmonic equation has to be solved with the appropriate boundaries. The method of solution when the required functions have been established is indicated.

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